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Tax evasion and confidence in institutions: Expectations models

Dissertation

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Contents

Introduction

The aim of this dissertation is the realization of mathematical models on tax evasion, more and more comprehensive and complex, from the (microeconomic) point of view of a taxpayer. In particular, we begin with a static model on tax evasion and confidence in institutions (that extends a classical model with maximization of expected utility), then we transform it in a model on the taxpayer's expectations (using stochastic variables and processes), finally we extend it to a stochastic control model with (public and private) investments.

The first model that we propose describes, in a static framework with maximization of expected well-being, the citizen's choice on declared income, taking into account his or her confidence in institutions and his or her sense of social responsibility.
The use of an economic model to describe the violation of the law, was proposed by Becker (1968). Along this line Allingham and Sandmo wrote a seminal paper on tax evasion (1972), modeling the fiscal choice as a gamble, in a microeconomic framework of expected utility maximization. But it is well known that the taxpayer's behavior is also driven by social, psychological and economic factors (among which, the trust in institutions) related to the tax morale (see Torgler, 2003). Furthermore, as it is clear in the detailed analysis of Rose-Ackerman (2004), the different causes of corruption in general, and the various possible ways to reduce tax evasion in particular, make taxes an economic topic characterized by a significant degree of complexity.

The second model that we propose, describes the taxpayer's expectations on the economic determinants of the fiscal choice and on his or her fiscal behavior in the future (it can be also seen as a forecasting model), both in discrete-time and in continuous-time, with the use of random variables and Monte Carlo methods.
Indeed, in the complex context of tax evasion, the taxpayer's expectations play a significant role (for example, in the policies oriented to raise tax compliance, or in the effect that a change of tax rate has in macroeconomic variables such as consumption, savings and investments). The importance

of expectations in determining economic choices was already emphasized by Keynes (1936) and it is possible to find in the literature different ways to model them. For instance adaptive expectations (described by Nerlove, 1958), that use the present and past information to forecast the future value of an economic variable (for example, a price). Or rational expectations (proposed by Muth, 1961, and used by Lucas in a general equilibrium model, 1972), that use the whole information provided by the model to forecast the expected value of a variable.

The third model that we propose concerns the taxpayer's expectations taking into account dynamical factors as public and private investments, within a stochastic control model.
The use of stochastic control models in economics began some decades ago (see, for instance, Bismut, 1975, and Merton, 1975). These models combine the instruments of stochastic calculus with those of deterministic control, in order to describe a controlled dynamical system under assumptions of uncertainty. In particular, stochastic control models allow to study the dynamics of (stochastic) state processes (describing, for example, the accumulation of income, capital or public expenditure), depending on (stochastic) control processes (describing, for example, consumption, savings or investments) that the economic agent chooses in order to maximize the (expected) total utility in a time interval.

In Chapter 1 we propose the static model on tax evasion with confidence in institutions. Chapter 2 deals with the model on taxpayer's expectations, both in discrete-time and in continuous-time. In Chapter 3 we present the stochastic control model on taxpayer's expectations with public and private investments. Then, there are the conclusions of the dissertation. Appendix A offers some specifications on technical and mathematical properties. In Appendix B there is a project of empirical verification for the assumptions of the static model. Appendix C introduces the code (contained in the attachments 1 and 2) of an algorithm used in Chapter 3.

Chapter 1

Static model

In this chapter we present a theoretical model of tax evasion, which takes into account the confidence of the taxpayer in respect of institutions and his or her sense of social responsibility. In order to do this, in addition to the utility of income, is attributed to the agent a utility ('confidence') of contributing to the collective welfare. Confidence is a function of declared income, tax rate and effectiveness of public expenditure.

The seminal model on tax evasion proposed by Allingham and Sandmo (1972) treats the choice of evading as a gamble. A citizen is called to pay taxes on his or her income W, at a fixed rate θ. If the citizen declares less than his or her income (i.e. he or she declares an amount $X < W$), it is assumed that he or she can be recognized as evader with a certain (perceived) probability p. In such case, the evader has to pay taxes on the concealed income $W - X$ at a penalty rate π, which is higher than the tax rate (i.e. he or she has to pay, overall, $\theta X + \pi(W - X)$). Taxpayer's marginal income utility is assumed to be positive and decreasing. Furthermore, the taxpayer chooses the declared income X so as to maximize the expected (with respect to the probability of being discovered) utility.

Yitzhaki (1974) modifies the model of Allingham and Sandmo applying a fine proportional to the evaded tax (rather than to the undeclared income). Namely, the discovered evader has to pay, overall, $\theta X + f\theta(W - X)$, with $f > 1$. In this way, the model fits to a widespread tax system. In the following, we denote the model of Allingham-Sandmo-Yitzhaki as the 'classical model'.

The classical model does not take into account the factors that make evasion different from a gamble. They are due to social and psychological motivations and are also connected with functioning of services and confidence in institutions (on various aspects of the tax morale see, for instance, the work of Torgler, 2003). Allingham and Sandmo themselves mention the possibility of using a more extensive form of utility, which takes into account a (binary type) variable of reputation as a citizen. However, they focus their analysis simply on the utility of income. These social, psychological and structural factors explain, among other things, the cases in which tax compliance is chosen despite an apparently convenient gamble. Moreover, in the classical model, if one assumes that the absolute risk aversion is decreasing with income, compliance increases as tax rate increases. But this relationship, as well as being counterintuitive, is contradicted by empirical studies in this regard (see, for instance, the work of Clotfelter, 1983, and that of Pommerehne and Weck-Hannemann, 1996). A comprehensive review of the literature on the empirical evidences in contrast to the classical model may be found in the introduction of the work of Myles and Naylor (1996) and in the survey of Freire-Seren and Panades (2013).

Various theoretical models have been developed to solve these contrasts. Cowell and Gordon (1988) propose a model in which utility depends (as well as on consumption) on public goods. Gordon (1989) proposes a model in which, to the utility of consumption, is added a (linear) term which decreases as hidden income increases. This added term can be interpreted as a psychic cost of evasion. The model of Bordignon (1993) includes a fairness constraint, which constitutes a ceiling on evasion. The fairness constraint depends on tax rate, public goods and behavior of the other citizens. The model of Myles and Naylor (1996) distinguishes the utility of evading (analogous to that in the classical model) from the utility of not evading (that takes into account income, social customs and behavior of the other citizens). Tax evasion occurs when the utility of evading is greater than the utility of not evading. In the model of Sour (2004), which is inspired by the models of Gordon (1989) and Myles and Naylor (1996), the utility function takes into account (besides income) psychic cost of evasion and behavior of the other citizens. Other models that in the utility function consider social norms and presence of the other citizens, are proposed by Kim (2003), Fortin, Lacroix, Villeval (2007), Dell'Anno (2009) and Traxler (2010).

In our model we assume the same hypotheses for the tax system made in the classical model. But unlike the classical model, the taxpayer chooses the declared income X so as to maximize the expected (with respect to the probability of being discovered p) 'well-being function' B, of the form

$$B = U + C\,,$$

where U is the classical utility of income and C is the 'confidence function' (a utility of contributing to the collective welfare).
For the use of the concept of well-being in the economical literature, see the survey of Stutzer and Frey (2010). For the statistical linkage between the

tax compliance attitude and the subjective well-being, see Helliwell (2003). A well-being function in which there are an addend concerning the economic utility and one for the moral utility, is proposed in the work of Parada Daza (2004). Well-being in our model can be considered a generalization of the utility in Gordon (1989), with the difference that the dependence on the hidden income may be non-linear (and with tax rate and effectiveness of public expenditure, not present in the form of the utility considered in Gordon (1989)). The confidence function can be interpreted as the social responsibility of the individual, or his or her confidence in institutions. This function depends, as well as on the declared income X, on the tax rate θ and on the perceived effectiveness of public expenditure α. For the role of trust in institutions in tax morale, see the works of Torgler (2003), Nabaweesi, Ngoboka, Nakku (2013), Birskyte (2014). For the use of the government effectiveness as an argument of the utility function, see Dell'Anno (2009).

In this chapter we show that, unlike the classical model, there are citizens who choose tax compliance even in case of convenient gamble. Moreover, in our model an increase in tax rate may lead to an increase in tax evasion. In regard to these issues, therefore, this model proposes a solution to the conflicts between the results of the classical model and the empirical findings. Furthermore, tax compliance is monotonically non-decreasing with respect to the effectiveness of public expenditure. This result is consistent with the econometric analysis of Schneider, Buehn, Montenegro (2010).

Within our model we can also describe different types of taxpayers, as the free rider, the honest citizen, the taxpayer sensitive to the effectiveness of public expenditure and the taxpayer sensitive to the tax rate. We offer an example in this regard.

In Section 1.1, our model is conceptually described and mathematically formalized. Section 1.2 deals with the analysis of the model. Section 1.3 offers some examples. In Section 1.4 there are the conclusions of this chapter.

1.1 The model

Let us imagine that a citizen with (positive) income W is called to pay taxes, at a (fixed) rate equal to θ, with $0 < \theta < 1$.
If the citizen declares his or her entire income, therefore, he or she pays a tax equal to θW. In case the citizen declares an amount $X < W$, it is assumed that he or she can be recognized as evader with (perceived) probability equal to p (with $0 < p < 1$).
In such case, the evader has to pay a fine $f > 1$ on the evaded tax $\theta(W - X)$, i.e. he or she has to pay $f\theta(W - X)$. For technical reasons, namely to avoid

dealing with the utility of negative amounts, we put a ceiling on the fine, assuming $f < \frac{1}{\theta}$ (see in this regard Appendix A.1). The latter hypothesis does not remove the interest of the model. For example, it is consistent with tax systems characterized by frequent checks and fines not too high.

According to our assumptions, the undiscovered evader ends up with an effective income Y equal to

$$Y = W - \theta X \,. \tag{1.1}$$

The discovered evader, instead, ends up with an effective income Z equal to

$$Z = W - \theta X - f\theta(W - X) \,. \tag{1.2}$$

As yet, our model follows the model of Allingham-Sandmo-Yitzhaki (except for the ceiling on the fine, not present in the classical model). Now, rather than limit ourselves to consider the utility of income (classical model), we assume that the citizen has his or her own 'well-being function' B of the form

$$B(y_1, y_2, \theta, \alpha) = U(y_1) + C(y_2, \theta, \alpha) \,,$$

where $y_1 \in]0, \infty[$ is the effective income, $y_2 \in]0, \infty[$ is the declared income, $\alpha \in]0, 1[$ is a parameter of (perceived) efficacy of public expenditure, the (real-valued) functions U and C are, respectively, the classical utility of income and the 'confidence' (a utility of contributing to the collective welfare). In this way we are modeling the fact that choices in terms of tax compliance are not a simple gamble (in which who evade wins if he or she is not discovered and loses otherwise), as in the classical model. They are also linked to the confidence of the citizen in respect of institutions and his or her sense of social responsibility and, in this way, to the well-being of the citizen (and therefore to social and psychological factors and to functioning of services).

The parameter α is subjective, since it refers to perceived effectiveness. It can be seen how a perception of quality of public services or policies (or even, for example, of control of corruption). One can obtain an indicator of α by statistical methods, such as in (Kaufmann-Kraay-Mastruzzi, 2010). The parameter α may also be interpreted as a perception of equity, or of distributive efficiency (for example, Pareto efficiency).

We assume that U has continuous second order derivative. We also assume that C has continuous second order derivatives with respect to all variables.
Implicitly, we can define $C(0, \theta, \alpha)$ and $C_{y_2}(0, \theta, \alpha)$ as limits (eventually not finished) for y_2 that tends to 0 from the right.

In order to have U increasing and strictly concave (risk aversion), we assume for all y_1

$$U'(y_1) > 0\,,$$

$$U''(y_1) < 0\,.$$

For C, with respect to the declared income, it is reasonable assuming similar hypotheses (admitting also the cases with C_{y_2} and $C_{y_2 y_2}$ equal to 0). Then, for all y_2, θ, α we have

$$C_{y_2}(y_2, \theta, \alpha) \geq 0\,,$$

$$C_{y_2 y_2}(y_2, \theta, \alpha) \leq 0\,.$$

We also assume that there exists $\overline{\theta} \in [0, \frac{1}{f}[$ such that for all $\theta > \overline{\theta}$ (and for all y_2, α) we have

$$C_{y_2 \theta}(y_2, \theta, \alpha) \leq 0\,,$$

namely, the marginal confidence of declared income is monotonically non-increasing with respect to θ, at least above $\overline{\theta}$. This assumption appears reasonable, because we imagine that if taxes seem too high, citizens lose confidence in institutions and are less motivated to give their fiscal contribution. Furthermore, we assume that for all y_2, θ, α we have

$$C_{y_2 \alpha}(y_2, \theta, \alpha) \geq 0\,,$$

that is, the marginal confidence of declared income is monotonically non-decreasing with respect to α. Also this assumption appears reasonable, because we imagine that if public expenditure is perceived as effective, the citizen acquires confidence in institutions and he or she is more motivated to give his or her fiscal contribution to the welfare of the community.
Notice that the classical model may be seen as a particular case of our model, for C identically zero (and, in general, for C constant with respect to y_2, because in these cases the marginal confidence C_{y_2} is identically zero).

1.2 Analysis and solution of the model

The expected well-being of a citizen who chooses to declare X is therefore equal to

$$E[B] = (1-p)B(W-\theta X, X, \theta, \alpha) + pB(W-\theta X - f\theta(W-X), X, \theta, \alpha)$$

$$= (1-p)U(W-\theta X) + pU(\,W-\theta X - f\theta(W-X)\,) + C(X,\theta,\alpha)\,.$$

We assume that the citizen declares an amount $\overline{X}$ that maximizes the expected well-being.
In case $\overline{X}$ is an interior maximum of $[0,W]$, it has to satisfy the following first order condition

$$\frac{\partial E[B]}{\partial X} = 0\,,$$

that is

$$-\theta(1-p)U'(W-\theta X) - (\theta - f\theta)pU'(\,W-\theta X - f\theta(W-X)\,) + C_{y_2}(X,\theta,\alpha) = 0\,, \tag{1.3}$$

that is

$$-\theta(1-p)U'(Y) + \theta(f-1)pU'(Z) + C_{y_2}(X,\theta,\alpha) = 0\,, \tag{1.4}$$

where Y and Z are defined, respectively, in (1.1) and (1.2).
The second order condition (sufficient for a relative interior maximum, once verified that of the first order) is given by

$$\frac{\partial^2 E[B]}{\partial X^2} < 0\,,$$

that is

$$\theta^2(1-p)U''(Y) + \theta^2(1-f)^2 pU''(Z) + C_{y_2 y_2}(X,\theta,\alpha) < 0\,. \tag{1.5}$$

This condition, thanks to the assumptions on U and C, is verified for each $X \in]0,W[$. It implies that $\frac{\partial E[B]}{\partial X}$ is decreasing with respect to X. Then $\frac{\partial E[B]}{\partial X}$ is null at a (unique) interior point of $[0,W]$ if and only if both the following conditions are verified

$$\frac{\partial E[B]}{\partial X}|_{X=0} > 0\,, \tag{1.6}$$

$$\frac{\partial E[B]}{\partial X}|_{X=W} < 0\,. \tag{1.7}$$

Therefore these constitute a necessary and sufficient condition for the existence of a (unique) interior point $\overline{X}$ of absolute maximum (figure 1.1, *case 1*).
If condition (1.6) is false, instead, the expected well-being is maximized for $\overline{X} = 0$ (because $\frac{\partial E[B]}{\partial X} < 0$ in $]0, W[$ and, thus, $E[B]$ is decreasing) (figure 1.1, *case 2*). If condition (1.7) is false, the expected well-being is maximized for $\overline{X} = W$ (because $\frac{\partial E[B]}{\partial X} > 0$ in $]0, W[$ and, thus, $E[B]$ is increasing) (figure 1.1, *case 3*).

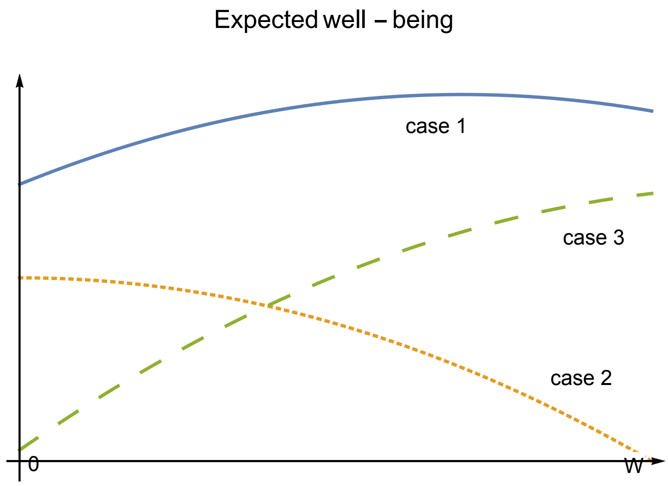

Figure 1.1: The expected well-being as a function of X.

Condition (1.6) may be explicitly written as follows (consider the left-hand side of (1.3) for $X = 0$)

$$-\theta(1-p)U'(W) - (\theta - f\theta)pU'(W - f\theta W) + C_{y_2}(0, \theta, \alpha) > 0\,,$$

that is equivalent (dividing by $-\theta U'(W - f\theta W)$) to

$$(1-p)\frac{U'(W)}{U'(W - f\theta W)} - (f-1)p - \frac{C_{y_2}(0,\theta,\alpha)}{\theta U'(W - f\theta W)} < 0\,,$$

that is

$$\frac{U'(W)}{U'(\,W(1-f\theta)\,)} < \frac{p(f-1)}{1-p} + \frac{C_{y_2}(0,\theta,\alpha)}{\theta U'(\,W(1-f\theta)\,)(1-p)}\,. \tag{1.8}$$

Condition (1.7) may be explicitly written as follows (consider the left-hand side of (1.3) for $X = W$)

$$-\theta(1-p)U'(W-\theta W)-(\theta-f\theta)pU'(W-\theta W)+C_{y_2}(W,\theta,\alpha)<0\,,$$

that is equivalent to

$$-\theta U'(W-\theta W)[1-p+(1-f)p]+C_{y_2}(W,\theta,\alpha)<0\,,$$

that is

$$[1-p+p-pf]-\frac{C_{y_2}(W,\theta,\alpha)}{\theta U'(W-\theta W)}>0\,,$$

that is equivalent to

$$pf<1-\frac{C_{y_2}(W,\theta,\alpha)}{\theta U'(\,W(1-\theta)\,)}\,. \tag{1.9}$$

In the classical model of Allingham-Sandmo-Yitzhaki, instead, the two conditions (1.8), (1.9) (obtained simply maximizing $E[U]$) may be explicitly written, respectively, as

$$\frac{U'(W)}{U'(\,W(1-f\theta)\,)}<\frac{p(f-1)}{1-p}\,,$$

and

$$pf<1\,.$$

Comparing the conditions of our model with those of the classical model, we realize that condition (1.8) is weaker than the first classical condition, while (1.9) is stronger than the second classical condition. Thus if null income is declared in our model, namely condition (1.8) is false, then the first classical condition is even false and therefore also in the classical model null income is declared.
On the other hand, if the entire income is declared in the classical model, namely the second classical condition is false, then condition (1.9) is even false and therefore also in our model the entire income is declared. In the following there is an example (see Section 1.3, Example 1) in which in our model the entire income is declared, whereas in the classical model there is tax evasion.

Now we examine how the parameter θ affects the interior point of absolute maximum.
Let $\overline{X}(\theta)$ be the solution of the first order condition (1.4), or (equivalently) of

$$(1-p)U'(Y)-(f-1)pU'(Z)-\frac{C_{y_2}(X,\theta,\alpha)}{\theta}=0\,. \tag{1.10}$$

Varying θ (and the relative $\overline{X}(\theta)$) in a suitable neighborhood, condition (1.10) remains verified (for details see Appendix A.2) and the left-hand side of (1.10) is a constant function (equal to 0) of θ, thus it has null derivative with respect to θ. On the basis of this property, differentiating the left-hand side of (1.10) respect to θ and setting

$$\overline{Y}=W-\theta\overline{X}\,,$$

$$\overline{Z}=W-\theta\overline{X}-f\theta(W-\overline{X})\,,$$

we obtain

$$(1-p)\Big(-\overline{X}-\theta\frac{\partial\overline{X}}{\partial\theta}\Big)U''(\overline{Y})+(1-f)p\Big(-\overline{X}-\theta\frac{\partial\overline{X}}{\partial\theta}-f(W-\overline{X})+f\theta\frac{\partial\overline{X}}{\partial\theta}\Big)U''(\overline{Z})$$

$$+\frac{1}{\theta^2}\Big[-\theta\Big(C_{y_2y_2}(\overline{X},\theta,\alpha)\frac{\partial\overline{X}}{\partial\theta}+C_{y_2\theta}(\overline{X},\theta,\alpha)\Big)+C_{y_2}(\overline{X},\theta,\alpha)\Big]=0$$

(for the existence of $\frac{\partial\overline{X}}{\partial\theta}$ see Appendix A.2), that is equivalent to

$$\frac{\partial\overline{X}}{\partial\theta}=\frac{-\theta}{D}\Big[\overline{X}\Big((1-p)U''(\overline{Y})-p(f-1)U''(\overline{Z})\Big)-pf(f-1)(W-\overline{X})U''(\overline{Z})+$$

$$+\frac{C_{y_2\theta}(\overline{X},\theta,\alpha)}{\theta}-\frac{C_{y_2}(\overline{X},\theta,\alpha)}{\theta^2}\Big] \tag{1.11}$$

(for the complete proof see Appendix A.2) where D is the left-hand side of the second order condition (1.5) evaluated at $\overline{X}$, that is

$$D=\theta^2(1-p)U''(\overline{Y})+\theta^2p(f-1)^2U''(\overline{Z})+C_{y_2y_2}(\overline{X},\theta,\alpha)\,.$$

Notice that the form of $\frac{\partial\overline{X}}{\partial\theta}$ in the classical model of Allingham-Sandmo-Yitzhaki is the one reported in the first line of equation (1.11). Assuming that the absolute risk aversion ($R_A(y_1)=-\frac{U''(y_1)}{U'(y_1)}$) is decreasing with income, one has $\frac{\partial\overline{X}}{\partial\theta}>0$ (see Yitzhaki, 1974, p.202), in contrast to the empirical results. In our case, the terms of the first line of equation (1.11) may assume values different than in the classical model, and the terms in the second line

of equation (1.11) are negative or null (at least for $\theta > \overline{\theta}$). For these reasons, in our model $\frac{\partial \overline{X}}{\partial \theta}$ can take negative values, consistently with the empirical results. In the following there is an example (see Section 1.3, Example 2) in this regard.
Already in (Gordon, 1989) the addition of a psychic cost changes the classical result (at the end of Example 2 it is shown as the utility of Gordon is a special case of our well-being function).

Let us analyze now the sensitivity with respect to the effectiveness of public expenditure (proceeding as before).
Differentiating the first member of (1.10) with respect to α (and setting null the derivative) we obtain

$$(1-p)\Big(-\theta\frac{\partial \overline{X}}{\partial \alpha}\Big)U''(\overline{Y}) - (f-1)p\Big(-\theta\frac{\partial \overline{X}}{\partial \alpha} + f\theta\frac{\partial \overline{X}}{\partial \alpha}\Big)U''(\overline{Z})$$

$$-\frac{\Big(C_{y_2y_2}(\overline{X},\theta,\alpha)\frac{\partial \overline{X}}{\partial \alpha} + C_{y_2\alpha}(\overline{X},\theta,\alpha)\Big)}{\theta} = 0\,,$$

that is

$$\frac{\partial \overline{X}}{\partial \alpha}\Big(-\theta(1-p)U''(\overline{Y}) - \theta p(f-1)^2U''(\overline{Z}) - \frac{C_{y_2y_2}(\overline{X},\theta,\alpha)}{\theta}\Big) = \frac{C_{y_2\alpha}(\overline{X},\theta,\alpha)}{\theta}\,,$$

that is equivalent to

$$\frac{\partial \overline{X}}{\partial \alpha} = -\frac{C_{y_2\alpha}(\overline{X},\theta,\alpha)}{D}\,.$$

Therefore, as it was reasonably expected, it is always

$$\frac{\partial \overline{X}}{\partial \alpha} \geq 0$$

(namely, tax compliance is monotonically non-decreasing with respect to the effectiveness of public expenditure).
This result is consistent with the econometric analysis of Schneider, Buehn, Montenegro (2010), where within a MIMIC model it occurs a negative relationship between the government effectiveness and the shadow economy (that can be a proxy of tax evasion).

1.3 A few examples

Example 1.
Let us see now an example in which in our model the entire income is declared, whereas in the classical model there is tax evasion. In this way, our model is consistent with the fact that there are citizens who do not evade despite tax evasion may arise as an apparently convenient gamble.
Let us consider a utility of income and a confidence function of the form

$$\begin{cases} U(y_1) = \ln y_1 \,, \\ C(y_2, \theta, \alpha) = \frac{\alpha}{\theta} \ln y_2 \,. \end{cases} \tag{1.12}$$

The assumptions on U and C are, therefore, all verified.
Assigning, in addition, the values

$$W = 1 \,, \; p = \frac{1}{5} \,, \; f = 3 \,, \; \theta = \frac{1}{5} \,, \; \alpha = \frac{1}{2} \,,$$

it results $f\theta < 1$ (the assumption about f is verified). Moreover $pf < 1$, namely, as we have seen previously, in the classical model there is tax evasion. Let us remember that condition (1.7) is explicitly written in condition (1.9)

$$pf < 1 - \frac{C_{y_2}(W, \theta, \alpha)}{\theta U'(\,W(1-\theta)\,)}$$

and, by using (1.12), it becomes

$$pf < 1 - \frac{\frac{\alpha}{\theta W}}{\frac{\theta}{W(1-\theta)}} \,,$$

that is

$$pf < 1 - \frac{\alpha(1-\theta)}{\theta^2} \,.$$

Since

$$1 - \frac{\alpha(1-\theta)}{\theta^2} = -9 \leq \frac{3}{5} = pf \,,$$

condition (1.7) is not verified and therefore, for this configuration of functions and parameters, in our model the entire income is declared.

Example 2.
As we have seen previously, in the classical model if the citizen has absolute risk aversion decreasing with income, then $\frac{\partial \overline{X}}{\partial \theta} > 0$, in contrast to the empirical studies.

Let us see now an example in which, instead, in our model a citizen has absolute risk aversion decreasing with income and, for certain values of the parameters, it is $\frac{\partial \overline{X}}{\partial \theta} < 0$ (thus, as taxes increase, declared income decreases). We assume the same utility and confidence functions (1.12) as in the previous example.
Notice that the absolute risk aversion of U is

$$R_A(y_1) = -\frac{U''(y_1)}{U'(y_1)} = -\frac{-\frac{1}{y_1^2}}{\frac{1}{y_1}} = \frac{1}{y_1}\,,$$

therefore $R_A(y_1)$ is decreasing with income. We assign the values

$$W = 1\,,\, p = \frac{1}{5}\,,\, f = 2\,,\, \theta = \frac{9}{20}\,,\, \alpha = \frac{1}{5}\,.$$

It results $f\theta < 1$ (the assumption about f is verified). In addition, condition (1.8) takes the form

$$\frac{\frac{1}{W}}{\frac{1}{W(1-f\theta)}} < \frac{p(f-1)}{1-p} + \frac{\lim_{y_2\to 0^+} \frac{\alpha}{\theta y_2}}{\frac{\theta(1-p)}{W(1-f\theta)}}\,,$$

then it is verified since the left-hand side is finite and $\lim_{y_2\to 0^+} \frac{\alpha}{\theta y_2} = +\infty$.
Condition (1.9) takes the form

$$pf < 1 - \frac{\alpha(1-\theta)}{\theta^2}\,,$$

and, for the configuration of parameters considered, it becomes equivalent to

$$1 < \frac{185}{162}\,.$$

Then there exists a unique X of interior maximum, obtainable as solution of the first order condition (in the equivalent form (1.10))

$$(1-\frac{1}{5}) \cdot \frac{1}{1-\theta X} + (1-2) \cdot \frac{1}{5} \cdot \frac{1}{1-\theta X - 2\theta(1-X)} + \frac{\frac{1}{5}}{\theta X} \cdot \frac{1}{-\theta} = 0\,,$$

that is equivalent to

$$4\Big(1-\theta X-2\theta(1-X)\Big)\theta^2 X-(1-\theta X)\theta^2 X-(1-\theta X)\Big(1-\theta X-2\theta(1-X)\Big) = 0\,.$$

For $\theta = \frac{9}{20}$ the solution is

$$\overline{X} = \frac{2}{585}(117 + \sqrt{26689})\,.$$

The sign of the derivative (1.11) coincides with that of the factor

$$\overline{X}\Big((1-p)U''(\overline{Y}) - p(f-1)U''(\overline{Z})\Big) - pf(f-1)(W-\overline{X})U''(\overline{Z})$$

$$+\frac{C_{y_2\theta}(\overline{X},\theta,\alpha)}{\theta} - \frac{C_{y_2}(\overline{X},\theta,\alpha)}{\theta^2}\,,$$

that is equal to -6.21297 (with numerical calculation), and therefore $\frac{\partial\overline{X}}{\partial\theta} < 0$.

Gordon (1989) provides another example of $\frac{\partial\overline{X}}{\partial\theta} < 0$ for a suitable choice of the parameters. This example may be cast in our setting by choosing a well-being function of the type

$$B(y_1, y_2) = U(y_1) - v(W - y_2)$$

where v is a non-negative constant. Notice that the confidence function is, in this case, indipendent of θ and α.

Figure 1.2 shows the curves of declared income as θ varies in the interval $[0.401, 0.499]$, in the classical model (lower curve) as well as in our model (upper curve), with the choice of functions and parameters of Example 2.
We can observe, in particular, that for $\theta = \frac{9}{20} = 0.45$ in the classical model it has positive derivative and in our model negative derivative, as we have seen in Example 2.
In addition, in the first part of the interval, we note that in our model the entire income is declared despite in the classical model there is tax evasion (convenient gamble), as it happens in Example 1 (but with a different choice of parameters).

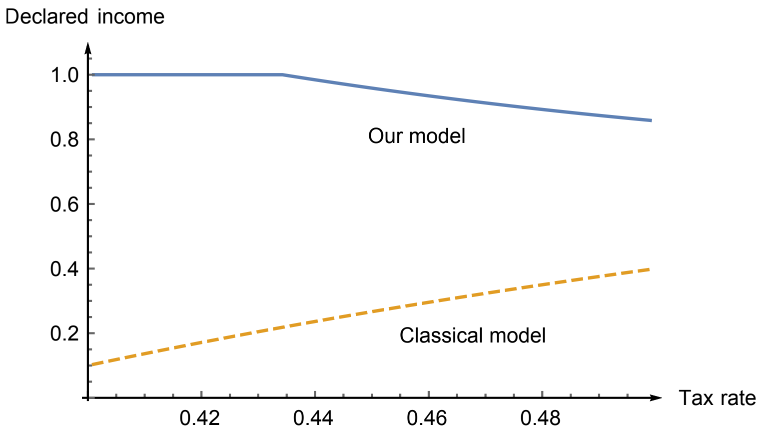

Figure 1.2: The declared income as a function of θ.

Example 3.
In this example we describe four types of taxpayers: the free rider, the honest citizen, the taxpayer sensitive to the effectiveness of public expenditure and the taxpayer sensitive to the tax rate.

a) *The free rider.*
Within our model, we can represent the free rider as a taxpayer that, in the decision on the income to declare, does not take into account his or her social responsibility. In other words, he or she maximizes the utility of income U, while his or her confidence function C is identically zero. In this way, the choice of the income to declare $\overline{X}$ matches exactly with that of the classical model.

To obtain an example of free rider, we assign, as in the previous example, the values

$$W = 1\,,\ p = \frac{1}{5}\,,\ f = 2\,,\ \alpha = \frac{1}{5}\,,$$

and we assume

$$\begin{cases} U(y_1) = \ln y_1\,, \\ C(y_2, \theta, \alpha) = 0\,. \end{cases}$$

Figure 1.3 shows the values of $\overline{X}$, as θ varies in the interval $[0.001, 0.499]$. Naturally, the curve of the classical model and that of our model are coincident.

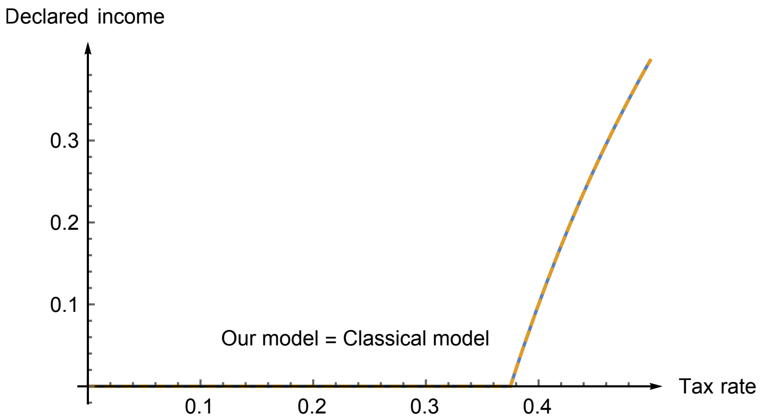

Figure 1.3: The free rider.

b) ***The honest citizen.***
Within our model, we can represent the honest citizen as a taxpayer that always chooses to declare the entire income.
In other words, his or her utility function U and his or her confidence function C are such that $\overline{X} = W$, for all W, p, f, θ, α.

To obtain an example of honest citizen, we assume

$$\begin{cases} U(y_1) = \ln(1 + y_1)\,, \\ C(y_2, \theta, \alpha) = 2y_2\,. \end{cases}$$

In this case, in fact, condition (1.9) becomes

$$pf < 1 - \frac{2}{\theta \frac{1}{1+W(1-\theta)}}\,.$$

In order to show that the condition is false for all choices of the parameters, it suffices to prove that

$$2 \geq \frac{\theta}{1 + W(1-\theta)}\,,$$

that is true, because $\theta < 1$ and $1 + W(1 - \theta) > 1$.
Figure 1.4, obtained with $W = 1$, $p = \frac{1}{5}$, $f = 2$, $\alpha = \frac{1}{5}$, shows the values of $\overline{X}$, as θ varies in the interval $[0.001, 0.499]$. In this case, in our model we have $\overline{X} = W$, despite in the classical model (where the taxpayer only maximizes U) we have $\overline{X} = 0$ in the whole interval.

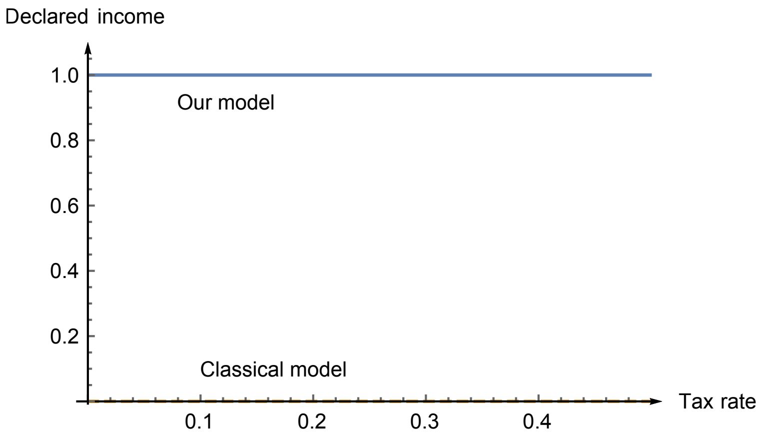

Figure 1.4: The honest citizen.

c) *The taxpayer sensitive to α.*
Figure 1.5, obtained with

$$\begin{cases} U(y_1) = \ln(y_1)\,, \\ C(y_2, \theta, \alpha) = \frac{1}{4}\frac{\alpha}{\theta}\ln y_2\,, \end{cases}$$

and

$$W = 1\,,\ p = \frac{1}{5}\,,\ f = 2\,,\ \theta = \frac{9}{20}\,,$$

shows the values of $\overline{X}$, as α varies in the interval $[0.001, 0.999]$.
It describes an example of taxpayer that is sensitive to α, namely a taxpayer that varies considerably his or her choice on the percentage of declared income depending on the perceived effectiveness of public expenditure.
In particular, he or she passes from a percentage lower than 30% (as in the classical model) for α near to 0, to a percentage equal to 100% (very far from that of the classical model) when α assumes high values.

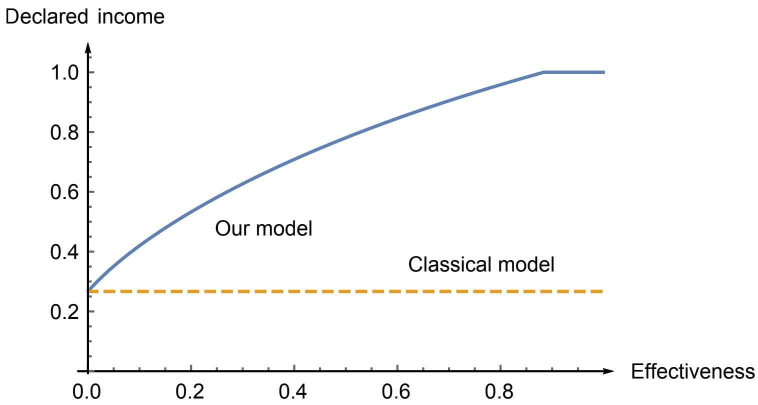

Figure 1.5: The taxpayer sensitive to α.

d) ***The taxpayer sensitive to*** θ**.**
Figure 1.6, obtained with

$$\begin{cases} U(y_1) = \ln(y_1)\,, \\ C(y_2, \theta, \alpha) = \frac{1}{2}\frac{\alpha}{\theta}\ln y_2\,, \end{cases}$$

and

$$W = 1\,,\ p = \frac{1}{5}\,,\ f = 2\,,\ \alpha = \frac{1}{5}\,,$$

shows the values of $\overline{X}$, as θ varies in the interval $[0.001, 0.499]$.
It describes an example of taxpayer that is sensitive to θ, namely a taxpayer that varies considerably his or her behavior depending on the tax rate.
In particular, the difference between the declared income in our model and the one declared in the classical model passes from a percentage equal to 100% for θ near to 0, to a percentage lower than 30% when θ assumes high values.

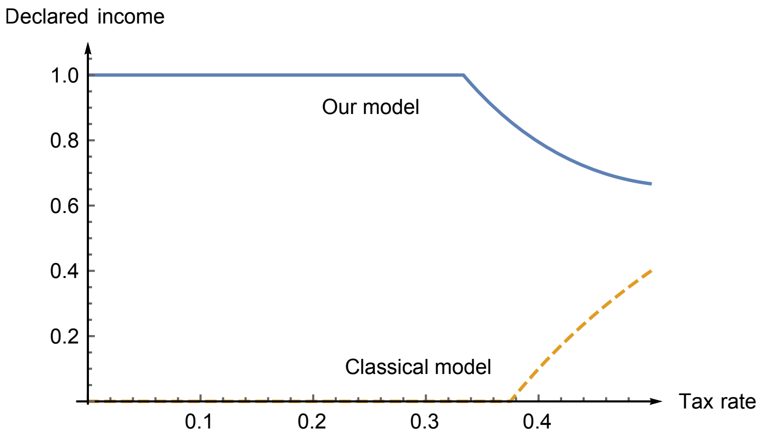

Figure 1.6: The taxpayer sensitive to θ.

1.4 Conclusions of this chapter

In this chapter we extend the classical theoretical models of tax evasion of Allingham-Sandmo (1972) and Yitzhaki (1974) by introducing the utility (confidence) of contributing to the collective welfare. In this framework, there are citizens who choose tax compliance even in case of convenient gamble. Moreover, we show that an increase in tax rate may lead to an increase in tax evasion. In regard to these issues, therefore, this model proposes a solution to the conflicts between the results of the classical model and the empirical findings. Within our model we can describe different types of taxpayers, as the free rider, the honest citizen, the taxpayer sensitive to the effectiveness of public expenditure and the taxpayer sensitive to the tax rate.

Naturally, the model that we have proposed is a simplification of a reality much more various and complex. It is also affected by the methodological problems that are common to all models based on the maximization of a form of expected utility. However, keeping in mind these limitations, it has the advantage of describing the choice of a taxpayer taking into account aspects not covered by the classical model, such as the confidence in institutions, the sense of social responsibility, the (perceived) fairness of tax rate and the (perceived) effectiveness of public expenditure.
In the Appendix B we propose a verification project for the assumptions of the model, with econometric and qualitative methods.

Possible extensions of the model may capture other aspects of tax systems and fiscal choices. For example, the assumption of fixed tax rate may be replaced by the assumption of progressive taxation. Moreover, the choice of a taxpayer may be influenced by the choices of other taxpayers.

In the following, we extend this model in a discrete-time and a continuous-time frameworks with random variables (Chapter 2) and in a stochastic control model (Chapter 3), in order to express the taxpayer's expectations (or, in an alternative interpretation, to forecast the taxpayer's behavior).

Chapter 2

Expectations models

It is known that the actions of economic agents in the present are based, as well as on the current economic variables, on the values that they expect for the variables in the future. In this sense, expectations play an important role in the models that describe the economic processes in a time interval (dynamical models).
In particular, in the framework of an economic model that takes into account the income tax, it can be important to understand what are the expectations of a taxpayer on the income that he or she will declare in the future.
The following describes two models, the first discrete-time (Section 2.1) and the second continuous-time (Section 2.2), on the expectations of a taxpayer (or, in an alternative interpretation, forecasting the behavior of a taxpayer). In other words, these models try to answer the question 'What are, today, the expectations of a taxpayer on the part of income that he or she will declare in the future?' and questions related to it (for instance, increasing or decreasing trend of evasion in absolute terms or in proportion to the income). To answer these questions, the models project into the future (using random variables and stochastic processes) the static model proposed in Chapter 1 (that extends the classical models in (Allingham and Sandmo, 1972) and (Yitzhaki, 1974), taking into account the confidence of the taxpayer in respect of institutions and his or her sense of social responsibility).

2.1 The discrete-time model

2.1.1 Case 1: only one choice

Let us consider, in the first instance, the discrete-time model with a single time interval (model with only one choice). t_0 is the 'time of expectations' (or forecasts) and t_1 is the 'time of choice'. For instance, $t_0 = 0$ can represent the present time and $t_1 = 1$ the time when, after a year, a citizen chooses the part of income to declare.

Let us imagine, retracing the model in Chapter 1, that the citizen after one year has a (positive) income W_1 and he or she is called to pay taxes at a rate (independent of income) θ_1, with $0 < \theta_1 < 1$. If he or she declares his or her entire income, therefore, he or she pays a tax of $\theta_1 W_1$. If he or she declares an amount $X_1 < W_1$, he or she can be recognized as evader with (perceived) probability equal to p_1 (with $0 < p_1 < 1$). In such case, he or she has to pay a fine $f_1 > 1$ on the evaded tax $\theta_1(W_1 - X_1)$, i.e. he or she has to pay $f_1\theta_1(W_1 - X_1)$ (as in Chapter 1, to avoid dealing with the utility of negative amounts, we put a ceiling on the fine, assuming $f_1 < \frac{1}{\theta_1}$. See in this regard Appendix A.1).

We assume, furthermore, that the citizen has its own 'well-being function' B of the form

$$B(y_1, y_2, \theta, \alpha) = U(y_1) + C(y_2, \theta, \alpha)\,,$$

with the same meaning and hypotheses of the previous chapter.

The citizen, at time t_1, declares an amount $\overline{X}_1$ that maximizes the expected well-being

$$E[B] = (1 - p_1)B(W_1 - \theta_1 X_1, X_1, \theta_1, \alpha_1) +$$

$$+ p_1 B(W_1 - \theta_1 X_1 - f_1\theta_1(W_1 - X_1), X_1, \theta_1, \alpha_1)\,.$$

From the analysis conducted in Chapter 1, based on the first and second order conditions, it results that there exists a (unique) interior point of absolute maximum, $\overline{X}_1 \in]0, W_1[$, if and only if both the conditions (1.8), (1.9) are verified. In addition, the expected well-being is maximized for $\overline{X}_1 = 0$ (null declaration) if and only if condition (1.8) is false. The expected well-being is maximized for $\overline{X}_1 = W_1$ (declaration of the entire income) if and only if condition (1.9) is false.

Therefore, the choice of declared income is a function of the parameters at time t_1. So we can indicate the declared income at time t_1 with $\overline{X}_1(W_1, \theta_1, \alpha_1, p_1, f_1)$.

But at time t_0 the citizen, in general, does not know exactly what value the parameters will assume at time t_1. For this reason, at time t_0 the parameters W_1 , θ_1 , α_1 , p_1 , f_1 can be modeled as random variables for which one can assume particular distributions.

The function $\overline{X}_1(W_1, \theta_1, \alpha_1, p_1, f_1)$, therefore, at time t_0 is itself a random variable, representing the citizen's expectations (or forecasts), at time t_0, on the income that he or she will declare at time t_1.
We can, at this point, raise the question of the existence of mean and variance (and higher order moments) for $\overline{X}_1$. In particular, the following proposition provides a sufficient condition in order that $\overline{X}_1$ has finite mean and finite variance (to indicate that the mean and the variance refer to the time t_0, they can be indicated by $E_0(\overline{X}_1)$ and $Var_0(\overline{X}_1)$).

Proposition 1. *If W_1 has finite mean and finite variance, then $\overline{X}_1$ also has finite mean and finite variance.*

Proof. As $0 \leq \overline{X}_1 \leq W_1$, we have $0 \leq E_0(\overline{X}_1) \leq E_0(W_1)$, therefore $\overline{X}_1$ has finite mean.
Moreover, as $0 \leq \overline{X}_1^2 \leq W_1^2$, we have $0 \leq E_0(\overline{X}_1^2) \leq E_0(W_1^2)$, whence

$$Var_0(\overline{X}_1) = E_0(\overline{X}_1^2) - (E_0(\overline{X}_1)\,)^2 \leq E_0(W_1^2) - (E_0(\overline{X}_1)\,)^2$$

$$= Var_0(W_1) + (E_0(W_1)\,)^2 - (E_0(\overline{X}_1)\,)^2\,,$$

therefore $\overline{X}_1$ has finite variance.

□

In the previous chapter is studied the dependence of the declared income on the tax rate as well as on the effectiveness of public expenditure, through the determination of $\frac{\partial \overline{X}_1}{\partial \theta_1}$ and $\frac{\partial \overline{X}_1}{\partial \alpha_1}$. Now we can ask other questions about the sensitivity to the characteristics of parameters. For instance, to understand how uncertainty on the income W_1 can determine uncertainty on the declared income $\overline{X}_1$, one can study the relationship between $Var_0(W_1)$ and $Var_0(\overline{X}_1)$. Naturally these analyses may be quite complex with respect to the general properties, but they can become easier if they refer to examples with the assignment of specific distributions (also using simulations and numerical algorithms).
In the following we propose some examples, in which we assume particular distributions for the parameters of the problem. With a Monte Carlo method the mean of $\overline{X}_1$, the variance and the histogram of the distribution are estimated (with the same procedure, also third, fourth and higher-order moments could be estimated). The main aim of these examples is the exposition of a general procedure (rather than the study of particular cases).
In the first example, the income W_1 has lognormal distribution and the other parameters are modeled as deterministic constants.

Example 4.
Let us consider a utility of income and a confidence function of the form

$$\begin{cases} U(y_1) = \ln y_1 \,, \\ C(y_2, \theta, \alpha) = \frac{\alpha}{\theta} \ln y_2 \,. \end{cases} \tag{2.1}$$

For the income W_1 we assume a lognormal distribution with parameters $-\frac{1}{2}\ln\frac{6}{5}$, $\sqrt{\ln\frac{6}{5}}$ (with this choice of parameters, the variable W_1 has mean 1 and variance 0.2).
We assume, in addition,

$$p = 0.15 \,,\; f = 1.6 \,,\; \theta = 0.45 \,,\; \alpha = 0.15 \,.$$

With a suitable algorithm n values of W_1 are simulated, for each of them the corresponding $\overline{X}_1$ is calculated (as a solution of the related optimization problem), and the arithmetic mean of these values

$$M = \frac{\sum_{j=1}^{n} \overline{X}_{1_j}}{n}$$

is an estimator of $E_0(\overline{X}_1)$. For the variance is used the unbiased estimator

$$S^2 = \frac{(\sum_{j=1}^{n} \overline{X}_{1_j}^2) - nM^2}{n-1} \,.$$

By using the central limit theorem and the properties of the standard normal distribution, it can be proved that with n simulations the random error is proportional to $\frac{1}{\sqrt{n}}$ and

$$I = \left[M - 1.96\sqrt{\frac{S^2}{n}}, M + 1.96\sqrt{\frac{S^2}{n}} \right]$$

is an approximate confidence interval for $E_0(\overline{X}_1)$, at the level of 95%. The algorithm produces also a histogram of the values $\overline{X}_{1_j}$.
Executing the algorithm with $n = 10^4$ simulations, it was obtained

$$M = 0.700789 \,,\; S^2 = 0.0976865 \,,\; I = [0.694663, 0.706915] \,,$$

with histogram in figure 2.1.
The economic meaning is that the taxpayer expects that after a year he or she will declare an income M, with an uncertainty expressed by the variance S^2 and by the histogram.
We can observe that the estimated variance for $\overline{X}_1$ is of the same order (about half) of the variance of W_1. In addition, the (deterministic) value of $\overline{X}_1$ in

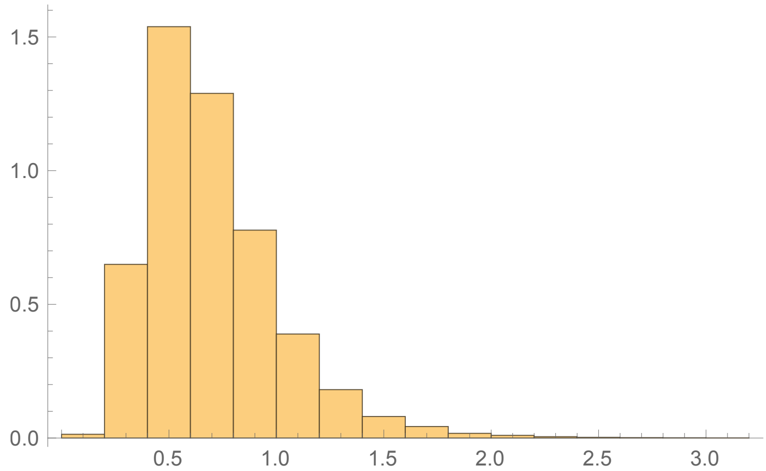

Figure 2.1: Estimated distribution of $\overline{X}_1$.

correspondence with $W_1 = 1$ (namely, for W_1 equal to its mean) is 0.703794. We can notice that this value is rather close to M, so as to belong to the confidence interval.
It should be recalled that the algorithms of this type, being based on the simulation of random variables, produce different results at each execution. For a more detailed study of a problem, therefore, the algorithm should be run several times, each time using a large number of simulations. _____

In the following example we proceed in a similar way, but this time assigning distributions of random variable also to the parameters p, f, α.

Example 5.

In this example we assume the same utility of income and confidence function (2.1) as in the previous example. Also for the income W_1 we assume the same distribution as in the previous example (lognormal with mean 1 and variance 0.2). We assume, in addition, $\theta = 0.45$.
But this time the parameters p, f, α are random variables with the following discrete distributions

$$\begin{cases} p = 0.1 & with\,probability\,\frac{1}{2}\,, \\ p = 0.2 & with\,probability\,\frac{1}{2}\,, \end{cases}$$

$$\begin{cases} f = 1.2 & with\,probability\,\frac{1}{2}\,, \\ f = 2 & with\,probability\,\frac{1}{2}\,, \end{cases}$$

$$\begin{cases} \alpha = 0.1 & with\,probability\,\frac{1}{2}\,, \\ \alpha = 0.2 & with\,probability\,\frac{1}{2}\,. \end{cases}$$

The algorithm is similar to that of the previous example, but this time are simulated n values of the random vector (W_1, p, f, α).
Executing the algorithm with $n = 10^4$ simulations, it was obtained (with the same notations of Example 4)

$$M = 0.699892\,,\ S^2 = 0.132088\,,\ I = [0.692769, 0.707016]\,,$$

with histogram in figure 2.2.

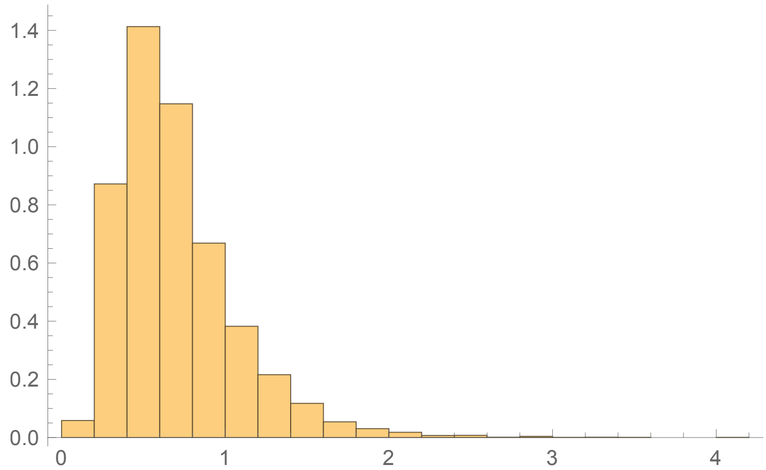

Figure 2.2: Estimated distribution of $\overline{X}_1$.

The interpretation, as before, is that the taxpayer expects that after a year he or she will declare an income M, with an uncertainty expressed by the variance S^2 and by the histogram.
We can observe that the estimated variance for $\overline{X}_1$ is greater than in the previous example, and this should be due to the fact that the uncertainty now, as well as on the income, is also on the probability of being discovered, the fine and the effectiveness of public expenditure.
In this case, the (deterministic) value of $\overline{X}_1$ in correspondence with the mean values $W_1 = 1$, $p = 0.15$, $f = 1.6$, $\alpha = 0.15$, is $\overline{X}_1 = 0.703794$. As in the previous example, it results rather close to M, so as to belong to the confidence interval.

Until now for simplicity we have not considered inflation. However, even assuming that the citizen keeps the same utility function over time, it is calibrated to a certain unit of measure. In other words it is reasonable that, if there is inflation, the marginal utility of a unit of money at time t_0 is different than that at time t_1.
For instance, let us assume that the value (adjusted for inflation) at time t_0 of a monetary amount v_1, received at time t_1, is $v_0 = v_1 e^{-\beta_1(t_1 - t_0)}$ (with

$\beta_1 > 0$ in case of inflation, $\beta_1 = 0$ in case of constant prices, $\beta_1 < 0$ in case of deflation. Eventually modeled as a random variable). Under this assumption, the expected well-being to maximize takes the form

$$E[B] = (1-p_1)B\Big(e^{-\beta_1(t_1-t_0)}(W_1-\theta_1 X_1), e^{-\beta_1(t_1-t_0)}X_1, \theta_1, \alpha_1\Big)+$$

$$+p_1 B\Big(e^{-\beta_1(t_1-t_0)}(\,W_1-\theta_1 X_1 - f_1\theta_1(W_1-X_1)\,), e^{-\beta_1(t_1-t_0)}X_1, \theta_1, \alpha_1\Big)\,.$$

The problem of existence and uniqueness of solution could be studied under this new assumption.
The following example is similar to Example 4, but this time we consider the presence of inflation (modeled with a random variable).
Example 6.
In this example we assume the same utility of income and confidence function (2.1) as in Example 4 and the same values for the parameters p, f, θ, α. For the income W_1 we assume a lognormal distribution with mean 1.1 and variance 0.2 (the corresponding parameters of distribution are $\ln\frac{11}{10} - \frac{1}{2}\ln(\frac{1}{5}e^{-2\ln\frac{11}{10}}+1)$, $\sqrt{\ln(\frac{1}{5}e^{-2\ln\frac{11}{10}}+1)}$). We assume, in addition, $t_0 = 0$, $t_1 = 1$, and for the variable e^{β_1} we assume the following discrete probability distribution:

$$\begin{cases} e^{\beta_1} = 1.05 \quad with\, probability\, \frac{1}{2}\,, \\ e^{\beta_1} = 1.15 \quad with\, probability\, \frac{1}{2}\,, \end{cases}$$

(to which corresponds a discrete distribution for β_1). The algorithm is similar to that of Example 4, but this time are simulated n values of the random vector (W_1, e^{β_1}).
Executing the algorithm with $n = 10^4$ simulations, it was obtained (with the same notations of the previous examples)

$$M = 0.778131\,,\; S^2 = 0.101089\,,\; I = [0.771899, 0.784362]\,,$$

with histogram in figure 2.3.
The interpretation, as before, is that the taxpayer expects that after a year he or she will declare an income M, with an uncertainty expressed by the variance S^2 and by the histogram. In this case the uncertainty is due not only to the income, but also to the two different values that the inflation can have at time 1.
Moreover, the (deterministic) value of $\overline{X}_1$ in correspondence with the mean values $W_1 = 1.1$, $e^{\beta_1} = 1.1$, is $\overline{X}_1 = 0.774173$. As in the previous examples, it results rather close to M, so as to belong to the confidence interval. $\_\_\_\_\_$

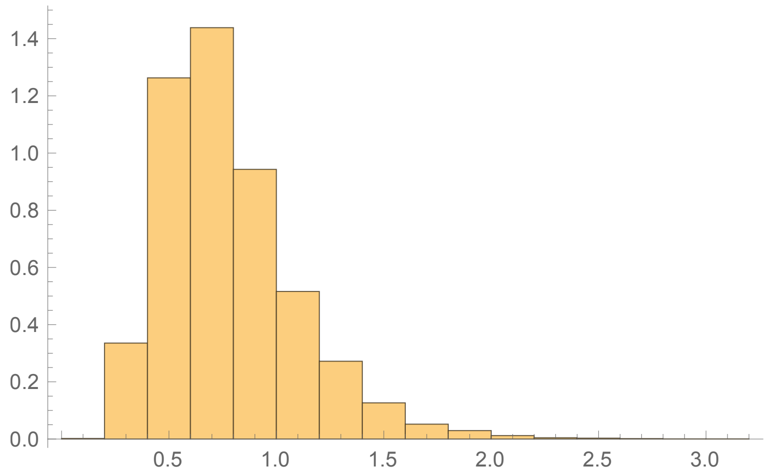

Figure 2.3: Estimated distribution of $\overline{X}_1$.

2.1.2 Case 2: n choices

Let us consider now the discrete-time model with n time intervals (model with n choices). t_0 is the 'time of expectations' (or forecasts) and $t_1, \ldots, t_n$ are the 'times of choices'. For instance, $t_0 = 0$ can represent the present time and $t_i = i$ the time when, after i years, a citizen chooses the part of income to declare for the year i.
Referring to the same model considered previously, the choice at time t_i depends only on the parameters at time t_i. In other words, solving for all i the optimization problem previously considered (for simplicity, without considering inflation), one obtains the values $\overline{X}_i(W_i, \theta_i, \alpha_i, p_i, f_i)$, each representing the declared income at time t_i (implicitly, we are assuming that the choices in different times are independent, namely that in every choice the consequences of the past and the effects in the future can be neglected).
The total declared income in the n years is, therefore, $\overline{X}_{Tot} = \sum_{i=1}^{n} \overline{X}_i$.

But at time t_0 the citizen, in general, does not know exactly what value the parameters will assume at the times t_i. For this reason, at time t_0 the parameters W_i, θ_i, α_i, p_i, f_i can be modeled as random variables for which one can assume particular distributions.
$\overline{X}_{Tot}$, therefore, at time t_0 is itself a random variable (function of the random variables corresponding to the various parameters in the different times) and it represents the citizen's expectations (or forecasts), at time t_0, on the total income that he or she will declare.
The following proposition provides a sufficient condition in order that $\overline{X}_{Tot}$ has finite mean $E_0(\overline{X}_{Tot})$ and finite variance $Var_0(\overline{X}_{Tot})$ (the proof is analogous to that of Proposition 1).

Proposition 2. *If the random variable* $W_{Tot} = \sum_{i=1}^{n} W_i$ *has finite mean and finite variance, then* $\overline{X}_{Tot}$ *also has finite mean and finite variance.*

We can also ask various questions about the sensitivity of $\overline{X}_{Tot}$ respect to the characteristics of parameters. For instance, to understand how uncertainty on parameters affects uncertainty on $\overline{X}_{Tot}$, one can study the relationship between the variances of the parameters and $Var_0(\overline{X}_{Tot})$. Also in this case, we can investigate these questions assuming specific distributions for the parameters (also using simulations and numerical algorithms).
In the following example we consider 5 time intervals (that can be interpreted as 5 years), assuming a lognormal distribution for the income (with mean and variance that are constant over time).

Example 7.
We are in the same setting of Example 4, but now we consider 5 time intervals. The algorithm is similar to that of the previous examples, but this time are simulated n values of the random vector $(W_1, W_2, W_3, W_4, W_5)$, and for each of them it determines the corresponding $\overline{X}_{Tot}$ (as sum of the X_i). As before, it estimates the mean of $\overline{X}_{Tot}$, the variance and a confidence interval at the level of 95%, and it produces a histogram of the values $\overline{X}_{Tot_j}$.
Executing the algorithm with $n = 10^4$ simulations, it was obtained (with the same notations of the previous examples)

$$M = 3.52116\,,\ S^2 = 0.499523\,,\ I = [3.5073, 3.53501]\,,$$

with histogram in figure 2.4.

Figure 2.4: Estimated distribution of $\overline{X}_{Tot}$.

The interpretation is that the taxpayer expects that in the next 5 years he or she will declare a total income M, with an uncertainty expressed by the variance S^2 and by the histogram.
In this case, the (deterministic) value of $\overline{X}_{Tot}$ in correspondence with the mean values $W_i = 1$ is 3.51897. As in the previous examples, it results rather close to M, so as to belong to the confidence interval. $\_\_\_\_\_$

Other examples might be obtained using, for W, different distributions at different times (for instance, lognormal with mean that increases over time) and varying over time the parameters p, f, θ, α (as deterministic or random variables).

2.2 The continuous-time model

Let us consider, now, the continuous-time model. t_0 is the 'time of expectations' (or forecasts) and $t \in]t_0, t_1]$ are the 'times of choices'. For instance, $t_0 = 0$ can represent the present time, $t_1 = 1$ the instant after a year. A possible interpretation is that a taxpayer (for instance, a dealer), in continuous time, chooses whether to provide (or not) a receipt and the correspondent amount.
Let us imagine (adapting to continuous time the previous model) that the citizen in the infinitesimal time interval $[t-dt, t]$ (with $t \in]t_0, t_1]$) has a (positive) income $w_t dt$ and he or she is called to pay taxes at a rate (independent of income) θ_t, with $0 < \theta_t < 1$.
If he or she declares $x_t < w_t$ (that is, $x_t dt$ represents the declared income in the infinitesimal interval $[t - dt, t]$), he or she can be recognized as evader with (perceived) probability equal to p_t (with $0 < p_t < 1$).
In such case, he or she has to pay a fine $f_t > 1$ on the evaded tax $\theta_t(w_t - x_t)dt$, i.e. he or she has to pay $f_t\theta_t(w_t - x_t)dt$ (as before, to avoid dealing with the utility of negative amounts we put a ceiling on the fine, assuming $f_t < \frac{1}{\theta_t}$).
According to our assumptions, therefore, the citizen ends up with an effective income (in the infinitesimal interval $[t - dt, t]$) equal to $y_t dt$, with

$$y_t = w_t - \theta_t x_t$$

if he or she is not discovered, and

$$y_t = w_t - \theta_t x_t - f_t\theta_t(w_t - x_t)$$

if he or she is discovered.
We assume, furthermore, that the citizen has its own 'well-being function' B of the form

$$B(y_t, x_t, \theta_t, \alpha_t) = U(y_t) + C(x_t, \theta_t, \alpha_t)\,,$$

for which we assume the same hypotheses of the discrete-time case.
The expected well-being of a citizen who at the instant t chooses to declare $x_t dt$ is, therefore, equal to

$$F(w_t, x_t, p_t, f_t, \theta_t, \alpha_t)$$

$$= (1 - p_t)B(w_t - \theta_t x_t, x_t, \theta_t, \alpha_t) + p_t B(w_t - \theta_t x_t - f_t\theta_t(w_t - x_t), x_t, \theta_t, \alpha_t)$$

$$= (1 - p_t)U(w_t - \theta_t x_t) + p_t U(\,w_t - \theta_t x_t - f_t\theta_t(w_t - x_t)\,) + C(x_t, \theta_t, \alpha_t)\,.$$

We assume that the citizen, at time t, declares an amount $\overline{x}_t dt$ that maximizes the expected well-being.
As in the discrete-time case, using the analysis in Chapter 1 one obtains conditions analogous to (1.8) and (1.9) which characterize the existence of internal solutions $\overline{x}_t \in]0, w_t[$ and the cases $\overline{x}_t = 0$ and $\overline{x}_t = w_t$.

The choice of declared income is, therefore, a function of the parameters at time t. So we can indicate the declared income at time t with $\overline{x}_t(w_t, \theta_t, \alpha_t, p_t, f_t)dt$.
Let us observe that, though we are dealing with a (continuous-time) dynamical model, the assumption that the choice is only a function of the parameters at time t allows us to bring back to the static case in Chapter 1.
The total declared income in the time interval $]t_0, t_1]$ is therefore (under suitable regularity hypotheses) $\overline{X}_{Tot} = \int_0^1 \overline{x}_t dt$.

But as in the discrete-time case, at time t_0 the citizen, in general, does not know exactly what value the parameters will assume at the times $t \in]0, 1]$. For this reason, at time t_0 the parameters w_t, θ_t, α_t, p_t, f_t can be modeled as random variables for which one can assume particular distributions.
$\overline{X}_{Tot}$, therefore, at time t_0 is itself a random variable (function of the random variables corresponding to the various parameters in the different times) and it represents the citizen's expectations (or forecasts), at time t_0, on the total income that he or she will declare.
The following proposition provides a sufficient condition in order that $\overline{X}_{Tot}$ has finite mean $E_0(\overline{X}_{Tot})$ and finite variance $Var_0(\overline{X}_{Tot})$ (we assume that the random variable in the statement, $\int_0^1 w_t dt$, is well defined, and this is true under suitable regularity hypotheses. The proof of the following proposition is analogous to that of Proposition 1).

Proposition 3. *If the random variable* $W_{Tot} = \int_0^1 w_t dt$ *has finite mean and finite variance, then* $\overline{X}_{Tot}$ *also has finite mean and finite variance.*

As in the discrete-time case, we can ask various questions about the sensitivity of $\overline{X}_{Tot}$ to the characteristics of parameters. For instance one can study the relationship between the uncertainty of the parameters and $Var_0(\overline{X}_{Tot})$. Naturally, also in this case, the analyses may be quite complex with respect to the general properties, but they can become easier if they refer to examples with the assignment of specific distributions (also using simulations and numerical algorithms).
The following example, of theoretical type, describes a continuous-time model, in which w_t follows the stochastic process of geometric Brownian motion, θ_t and α_t are deterministic straight lines (the model, under these assumptions, is formalized as a stochastic control problem).
Example 8.
In order to make a first example, in the time interval $[0, 1]$, we consider a taxpayer with a utility of income U and a confidence function C, and we assume that his or her income follows a geometric Brownian motion w_t with percentage drift m and percentage volatility σ. We assume, in addition, that the tax rate θ_t and the (perceived) effectiveness of public expenditure α_t vary over time as deterministic straight lines, with slope (respectively) m_θ, m_α, and that p and f are deterministic constants.
The economic meaning is that the taxpayer expects, for his or her income, an aleatory behavior (expressed by the geometric Brownian motion). He or she, furthermore, expects that the tax rate and the effectiveness of public expenditure (varying linearly over time) have an uncertainty negligible compared to that of income, so not to be considered in the model.
In this setting, the model may be formalized (here we do not discuss the analytical hypotheses of regularity) as the following stochastic control problem (with constraints):

$$\sup_{x_t\,(with\,0\le x_t\le w_t)} E_0\Big(\int_0^1 F(w_t, x_t, \theta_t, \alpha_t)dt\Big)\,,$$

under the conditions

$$dw_t = mw_t dt + \sigma w_t dB_t\,,$$

$$d\theta_t = m_\theta dt\,,$$

$$d\alpha_t = m_\alpha dt\,,$$

with initial values w_0, θ_0, α_0. B_t is a Brownian motion, the coefficients m, σ, m_θ, m_α are deterministic constants. $E_0()$ represents the mean with respect to the available information at time 0. To be consistent with the hypotheses, we assume $w_0 > 0$, $0 < \alpha_0 < 1$, $0 < \alpha_0 + m_\alpha < 1$, $0 < \theta_0 < \frac{1}{f}$, $0 < \theta_0 + m_\theta < \frac{1}{f}$.
It is interesting to observe (heuristically) that the Hamilton-Jacobi-Bellman (HJB) equation (stochastic case with constraints)

$$\begin{aligned}\sup_{x\,(with\,0\leq x\leq w)} \quad & \Big\{V_t(w,\theta,\alpha,t) + V_w(w,\theta,\alpha,t)mw + V_\theta(w,\theta,\alpha,t)m_\theta \\ & +V_\alpha(w,\theta,\alpha,t)m_\alpha + \tfrac{1}{2}V_{ww}(w,\theta,\alpha,t)\sigma^2 w^2 + F(w,x,\theta,\alpha)\Big\} = 0\end{aligned}$$

(with terminal condition $V(w,\theta,\alpha,1) = 0$, and V value function of the problem) returns the (static) maximization problem relative to the instant t

$$\sup_{x\,(with\,0\leq x\leq w)} \Big\{F(w,x,\theta,\alpha)\Big\}\,.$$

In the next chapter we study (with a rigorous analysis) stochastic control problems of this type, with additional hypotheses that do not allow to bring them back to the static case. -----

The next example is obtained from the previous one, by assigning the values of the parameters. In this case the problem can be solved numerically, discretizing the time interval. As in the discrete-time case, the main aim of this example is the exposition of a general procedure (rather than the study of a particular case).

Example 9.
In this example we assume the same utility of income and confidence function (2.1) as in Example 4 and the same values for the parameters p, f (constant over time). In addition, as in Example 8, we assume that the income follows a geometric Brownian motion w_t in the time interval $[0,1]$. Moreover we assign values to the parameters of the geometric Brownian motion, namely $m = 0.2$, $\sigma = 1$, $w_0 = 1$. For simplicity, we assume that θ_t and α_t are constant (that is, with the notations of the previous example, $m_\theta = m_\alpha = 0$), with the same values of Example 4.
In the algorithm we discretize the time interval $[0,1]$ in 10 equally spaced sub-intervals. We simulate $10 \times n$ random variables with standard normal distribution and, thus, we generate n discretized paths of the geometric Brownian motion w_t. By solving numerically the static optimization problem, we determine n (discretized) paths of $\overline{x}_t$. Finally, by means of numerical integration, for each path of $\overline{x}_t$ we determine the corresponding $\overline{X}_{Tot}$. As in

the previous examples, we estimate the mean of $\overline{X}_{Tot}$, the variance and a confidence interval at the level of 95%, and we provide a histogram of the values $\overline{X}_{Tot_j}$.
Executing the algorithm with $n = 10^4$ simulations of paths, we get (with the same notations of the previous examples):

$$M = 0.77847\,,\ S^2 = 0.0455622\,,\ I = [0.774286, 0.782654]\,,$$

with histogram in figure 2.5.

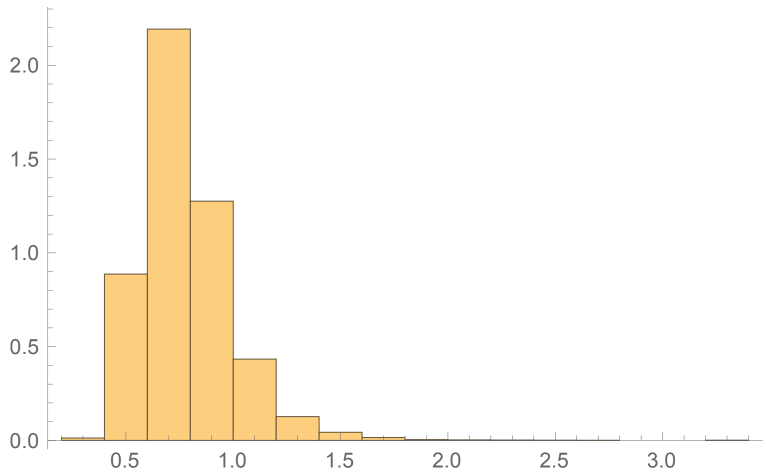

Figure 2.5: Estimated distribution of $\overline{X}_{Tot}$.

With regard to the interpretation, M represents the taxpayer's expectation on the total income that he or she will declare in a unitary period of time (for instance, one month), in which he or she will make some choices (for instance, about receipts). The variance S^2 and the histogram describe the taxpayer's uncertainty.
The (deterministic) value of $\overline{X}_{Tot}$ in correspondence with w_t deterministic ($\sigma = 0$), calculated with numerical integration, is 0.779109. Also in this case, it results rather close to M, so as to belong to the confidence interval.

Other examples can be obtained modeling w_t as another type of stochastic process (for instance, mean reverting jump) and varying over time the parameters p, f, θ, α (as deterministic or random variables).

We may also take into account the presence of inflation, by maximizing the expected well-being

$$F(w_t, x_t, p_t, f_t, \theta_t, \alpha_t, \beta_t) = (1-p_t)B\Big(e^{-\beta_t(t-t_0)}(w_t-\theta_t x_t), e^{-\beta_t(t-t_0)}x_t, \theta_t, \alpha_t\Big)+$$

$$+p_t B\Big(e^{-\beta_t(t-t_0)}(\,w_t-\theta_t x_t-f_t\theta_t(w_t-x_t)\,),e^{-\beta_t(t-t_0)}x_t,\theta_t,\alpha_t\Big)\,,$$

where β_t (eventually, modeled as a random variable) determines, as in the discrete-time case, the inflation rate. New examples could be obtained under this assumption of inflation.

2.3 Conclusions of this chapter

We have seen two models, the first discrete-time and the second continuous-time, on the expectations of a taxpayer (or, in an alternative interpretation, forecasting the behavior of a taxpayer). The models have been described and studied in their main properties, also some computational examples have been provided.
A possible development is the use of these models as a part of a general equilibrium model with taxpayer's expectations.
In these models the parameters vary over time and they are described by random variables. However, by construction, at each instant of time the choice of the taxpayer is brought back to an optimization problem of the type studied in Chapter 1. In other words, to solve these models one can refer to a static problem.
In other cases, however, it is useful to consider problems not referable to the static case. For instance, let us consider the following paradox. Slightly weakening the hypotheses on the utility U, we consider the case of a taxpayer with linear utility (that describes risk neutrality) $U(y_1) = ky_1$ (with $k > 0$) and confidence function identically zero $C(y_2, \theta, \alpha) = 0$. It is easy to demonstrate (the problem is studied in detail in the next chapter) that, in the case $pf = 1$, all the choices of declared income $\overline{X} \in [0, W]$ are equivalent (in other words, there is a paradoxical situation in which the risk neutral taxpayer, in this case, has no criteria for choosing). In the next chapter this paradox is solved extending to a dynamical model.
More in general, in the next chapter we study (continuous-time) models that, in the choices of the taxpayer, take into account public and private investments. These models are presented as stochastic control problems, not referable to the static case.

Chapter 3

Stochastic control model

In this chapter we present a class of stochastic control models, that describe the expectations of a taxpayer (or, in an alternative interpretation, forecasting the behavior of a taxpayer) on the part of income that he or she will declare in the future.
In particular we model a taxpayer who takes into account public and private investments. Indeed, a part of the collective tax revenue is invested by the public authorities and it raises the future income of the taxpayer, who is more motivated to tax compliance (indirect effect). Furthermore, a part of the taxpayer's effective income is invested by himself or herself, and it raises his or her future income. Thus, the taxpayer's compliance decisions take into account the consequences of the past and the effects in the future, and the problem is not referable to the static case (unlike the previous chapter).

In the literature there are various (continuous-time) control models with tax evasion (also taking into account dynamics of investment). See the works of Lin and Yang (2001) (with the comment of Dzhumashev and Gahramanov, 2011), Chen (2003), Dzhumashev (2007), Dzhumashev and Gahramanov (2008), Cerqueti and Coppier (2011), Levaggi and Menoncin (2013), Célimène et al. (2014).
Our model has various points in common with the previous literature, such as a tax system in which the discovered evader has to pay a fine, the declared income as control process, the presence of public and private investments. But our model has different assumptions and structure, because we aim at describing the expectations of a taxpayer, which maximizes (in a time-interval) his or her expected well-being (sum of the utility of income and the confidence in institutions, as in the previous chapters). We assume that the taxpayer's income follows a stochastic process depending on public and private investments.

In Section 3.1, the general model is presented. In Section 3.2, the paradox with risk neutrality, that we have seen before, is formalized and solved. Section 3.3 deals with a class of models for which we study (by means of a transformation of variables) the analytical properties, and provides a computational example. In Section 3.4 there are the conclusions of this chapter.

3.1 General model

We consider the problem of a taxpayer who has to decide his or her compliance (the process x_t) in order to maximize the following functional

$$E_0\Big(\int_0^T F(w_t, x_t)dt\Big)\,,$$

where the taxpayer's income evolves in time according to the stochastic differential law

$$dw_t = g(x_t, w_t)dt + h(w_t)dB_t\,, \tag{3.1}$$

with initial value $w_0 > 0$. B_t is a Brownian motion, in a probability space $(\Omega, \mathscr{F}, P)$, with augmented natural filtration $\{\mathscr{F}_t\}$ (that is, $\{\mathscr{F}_t\}$ is obtained adding the negligible events of $\mathscr{F}$ to the σ-algebras of the filtration generated by the Brownian motion).
$E_0()$ is the conditional expectation with respect to the initial value w_0. $F(w_t, x_t)$ is the expected well-being of the taxpayer, that is

$$F(w_t, x_t) = (1-p)U(w_t - \theta x_t) + pU(\,w_t - \theta x_t - f\theta(w_t - x_t)\,) + C(x_t, \theta, \alpha)\,,$$

as described in the continuous-time model of Section 2.2. However, this time we assume that w_t follows the stochastic differential law (3.1), namely, the income has a trend driven by the function g and an uncertainty driven by the function h.
The interpretation is that the taxpayer expects that in the time interval $[0, T]$ he or she will declare a (stochastic) total income $\overline{X}_{Tot} = \int_0^T \overline{x}_t dt$, where $\overline{x}_t$ ($t \in [0, T]$) is the optimal control process of the problem.
As before, the model describes the expectations of the taxpayer on the part of income that he or she will declare in the future. But now, the taxpayer's compliance decision becomes truly dynamical, because the trend function g depends on x_t (that is, the choice at time t determines the trend of the income in the future). For instance (as we can see in the sections 3 and 4) the function g can express the effect of investments (public and private) on the income.

As before, we assume for the parameters the hypotheses $0 < p < 1$, $f > 1$, $0 < \alpha < 1$, $0 < \theta < \frac{1}{f}$. For the functions U, C and their derivatives, we assume the same hypotheses of Section 1.1. Furthermore, we assume that the (real) functions g, h are measurable (with respect to the real Borel σ-algebras).
We assume, also, that the set of admissible control processes is

$$A = \left\{ \begin{array}{l} x(t,\omega) : [0,T] \times \Omega \to \mathbb{R}\, such\, that : \\ 1)\, x_t(\omega)\, is\, measurable\, with\, respect\, to\, \mathscr{F}_t\, and\, \mathscr{B}(\mathbb{R}),\ for\, all\, t. \\ 2)\, There\, exists\, a\, Markov\, process\, w\, solution\, of\ (3.1), \\ pathwise\, unique\, and\, positive. \\ 3)\, 0 \le x_t \le w_t\, (in\, [0,T] \times \Omega). \\ 4)\, E_0\Big(\int_0^T F(w_t, x_t) dt \Big)\, exists. \end{array} \right\}.$$

The value function of the problem takes the form

$$V(w_t, t) = \sup_{x \in A} E_t \Big(\int_t^T F(w_s, x_s) ds \Big)$$

(with $E_t()$ the conditional expectation with respect to the initial value w_t).

3.1.1 Heuristic derivation of the HJB equation

In the following we obtain the HJB equation as an heuristic necessary condition, under suitable conditions of regularity (for a study on the hypotheses that assure the validity of the HJB equation, see Krylov, 1980).
We can write Bellman's principle (for $0 \le h \le T - t$) as

$$V(w_t, t) = \sup_{x \in A} E_t \Big[\int_t^{t+h} F(w_s, x_s) ds + V(w_{t+h}, t+h) \Big]$$

(see Fleming and Soner, 2006, p. 132), that is

$$\sup_{x \in A} E_t \Big[\int_t^{t+h} F(w_s, x_s) ds + V(w_{t+h}, t+h) - V(w_t, t) \Big] = 0\,,$$

and, dividing by h,

$$\sup_{x \in A} \Big[E_t \Big(\frac{\int_t^{t+h} F(w_s, x_s) ds}{h} \Big) + E_t \Big(\frac{V(w_{t+h}, t+h) - V(w_t, t)}{h} \Big) \Big] = 0\,.$$

In the hypotheses that we can use Ito's formula for $V(w_t,t)$, namely $dV(w_t,t) = \Big(V_w(w_t,t)g(x_t,w_t) + \frac{1}{2}V_{ww}(w_t,t)(h(w_t))^2 + V_t(w_t,t)\Big)dt + V_w(w_t,t)h(w_t)dB_t$, it results

$$\begin{aligned}\sup_{x\in A} \quad &\Big[E_t\Big(\tfrac{\int_t^{t+h} F(w_s,x_s)ds}{h}\Big)\\ &+E_t\Big(\tfrac{\int_t^{t+h}\big(V_w(w_s,s)g(x_s,w_s)+\frac{1}{2}V_{ww}(w_s,s)(h(w_s))^2+V_t(w_s,s)\big)ds}{h}\Big)\\ &+E_t\Big(\tfrac{\int_t^{t+h} V_w(w_s,s)h(w_s)dB_s}{h}\Big)\Big] = 0\end{aligned}$$

and, if $E_t\Big(\int_t^{t+h} V_w(w_s,s)h(w_s)dB_s\Big) = 0$ (as it happens, for instance, in the case $E\Big(\int_t^{t+h}\big(V_w(w_s,s)h(w_s)\big)^2 ds\Big) < +\infty$, for the properties of conditional expectations and stochastic integrals, with $E()$ the mean in the probability space of the Brownian motion), we have

$$\begin{aligned}\sup_{x\in A} \quad &\Big[E_t\Big(\tfrac{\int_t^{t+h} F(w_s,x_s)ds}{h}\Big)\\ &+E_t\Big(\tfrac{\int_t^{t+h}\big(V_w(w_s,s)g(x_s,w_s)+\frac{1}{2}V_{ww}(w_s,s)(h(w_s))^2+V_t(w_s,s)\big)ds}{h}\Big)\Big] = 0\,.\end{aligned}$$

In the hypotheses that we can take the limit, for $h \to 0^+$, under the supremum and the conditional expectation, it results

$$\sup_{x\in A}\Big[E_t\Big(F(w_t,x_t)\Big)+E_t\Big(V_w(w_t,t)g(x_t,w_t)+\frac{1}{2}V_{ww}(w_t,t)(h(w_t))^2+V_t(w_t,t)\Big)\Big] = 0\,,$$

that is

$$\sup_{x\in A}\Big[F(w_t,x_t) + V_w(w_t,t)g(x_t,w_t) + \frac{1}{2}V_{ww}(w_t,t)(h(w_t))^2 + V_t(w_t,t)\Big] = 0\,.$$

Therefore, the HJB equation for our stochastic control problem is

$$\sup_{x\,(with\,0\le x\le w)}\Big[F(w,x) + V_w(w,t)g(x,w) + \frac{1}{2}V_{ww}(w,t)(h(w))^2 + V_t(w,t)\Big] = 0\,.$$

We can also assume, by definition of V, the (terminal) condition at time T

$$V(w,T) = 0\,.$$

3.2 Solution of the paradox on risk neutrality

Let us consider the static and deterministic model in Chapter 1. Slightly weakening the hypotheses on the utility U (in order to consider the linear case), we can assume

$$\begin{cases} U(y_1) = ky_1 \,, \\ C(y_2, \theta, \alpha) = 0 \,, \end{cases}$$

with $k > 0$. Namely, the taxpayer has linear utility (that describes risk neutrality). Being the confidence identically zero, this case can also be described within the classical models of (Allingham and Sandmo, 1972) and (Yitzhaki, 1974) (weakening the hypotheses on U).
In particular, let us assume $pf = 1$. In this case, the expected well-being takes the form

$$(1-p)k(W - \theta X) + pk(\, W - \theta X - f\theta(W - X)\,) = (1-\theta)kW \,,$$

that is independent of X. For this reason, in this case all the choices of declared income $\overline{X} \in [0, W]$ are equivalent for the taxpayer. Namely, there is a paradoxical situation in which the risk neutral taxpayer, in this case, has no criteria for choosing (we have already outlined this paradox in the conclusions of the previous chapter).
In order to solve the paradox, we put the choice of the taxpayer in a dynamical context with the presence of public investments, considering the (deterministic) control problem

$$\sup_{x_t \,(with\, 0 \leq x_t \leq w_t)} \int_0^1 F(w_t, x_t) dt \,,$$

$$dw_t = [m + c\ln(1 + \frac{x_t}{w_t})]dt \,,$$

with

$$F(w_t, x_t) = (1-p)k(w_t - \theta x_t) + pk(\, w_t - \theta x_t - f\theta(w_t - x_t)\,) = (1-\theta)kw_t \,,$$

$c > 0$, $w_0 > 0$, $m > -w_0$. This is a deterministic case (weakening the hypotheses on U) of the general control model that we have described in Section 3.1. The function $c\ln(1 + \frac{x_t}{w_t})$, in the deterministic differential law, expresses the effect of public investments on the trend of the income (to be more precise, it is an indirect effect, expressing that the taxpayer is motivated

to declare his or her income by the (expected) positive effect of public investments). The constant m expresses the other factors that drive the expected trend of the income.
It is easy to see that the solution of the problem is $\overline{x}_t = w_t$, for all $t \in [0,1]$ (because this choice of the control function x_t maximizes the state function w_t, see the elementary comparison result in Appendix A.3, and in this case the expected well-being F is (strictly) increasing with respect to w_t). The economic interpretation is that, now, the choices are not equivalent, because the taxpayer takes into account the effect of public investments.
We can also complete the analytical study of the problem, considering the value function

$$V(w_t,t) = \int_t^1 (1-p)k(w_s - \theta w_s) + pk(\,w_s - \theta w_s - f\theta(w_s - w_s)\,)ds$$

$$= \int_t^1 k(1-\theta)w_s ds\,.$$

The income takes the form $w_t = w_0 + [m + c\ln(2)]t$ (due to its differential law, with $x_t = w_t$), with primitive function $w_0 t + \frac{m+c\ln(2)}{2}t^2$. Thus we can write the value function as

$$V(w_t,t) = k(1-\theta)\left[w_0 + \frac{m+c\ln(2)}{2} - w_0 t - \frac{m+c\ln(2)}{2}t^2\right],$$

that is, with $w_0 = w_t - [m + c\ln(2)]t$,

$$V(w_t,t) = k(1-\theta)\left[w_t(1-t) + \frac{m+c\ln(2)}{2}(1-t)^2\right].$$

It is easy to see that it is solution of the HJB equation

$$V_t(w,t) + V_w(w,t)[m + c\ln(2)] + k(1-\theta)w = 0\,,$$

with the terminal condition

$$V(w,1) = 0\,.$$

3.3 Models with investment coefficients

Let us consider a particular case of the general model that we have seen in Section 3.1, with utility of income and confidence function of the form

$$\begin{cases} U(y_1) = a\ln(y_1)\,, \\ C(y_2,\theta,\alpha) = b\frac{\alpha}{\theta}\ln(l+y_2)\,, \end{cases}$$

and with

$$\begin{cases} g(x_t,w_t) = [m + c_1\frac{x_t}{w_t} + c_2\frac{\overline{y}_t}{w_t}]w_t\,, \\ h(w_t) = \sigma w_t\,, \end{cases}$$

where

$$\overline{y}_t = (1-p)(w_t - \theta x_t) + p(\,w_t - \theta x_t - f\theta(w_t - x_t)\,)$$

(with the meaning of the previous chapter. We assume for p, f, α, θ the same hypotheses of Section 3.1). In this case, the expected well-being can be expressed by

$$F(w_t,x_t) = (1-p)a\ln(w_t-\theta x_t)+pa\ln(\,w_t-\theta x_t-f\theta(w_t-x_t)\,)+b\frac{\alpha}{\theta}\ln(l+x_t)\,.$$

In this way, we have defined the class (varying the parameters $a > 0$, $b \geq 0$, m, $c_1 \geq 0$, $c_2 \geq 0$, $\sigma \neq 0$, $l > 0$, $w_0 > 0$) of stochastic control problems

$$\sup_{x\in A} E_0\Big(\int_0^T F(w_t,x_t)dt\Big)\,,$$

$$dw_t = \Big[m + c_1\frac{x_t}{w_t} + c_2\frac{\overline{y}_t}{w_t}\Big]w_t dt + \sigma w_t dB_t\,, \tag{3.2}$$

with initial value $w_0 > 0$.

Notice that the assumptions of Section 1.1, on U and C, are verified. The constant l occurs in the confidence function for technical reasons. It can also be chosen very small, for instance $l = 10^{-6}$.

The function $c_1\frac{x_t}{w_t}$ expresses the effect of public investments on the trend of the income (as we have seen before, it is an indirect effect, expressing that the taxpayer is motivated to declare his or her income by the (expected) positive effect of public investments). The function $c_2\frac{\overline{y}_t}{w_t}$ expresses the effect of private investments on the trend of the income (the effect is linked to the effective income, through $\overline{y}_t$). We can indicate the weights c_1, c_2, respectively, as public and private investment coefficients. The constant m expresses the other factors that drive the expected trend of the income. We have chosen the linear form for the investments, but also other functional forms can be chosen (for instance, the logarithmic form $c_1\ln(1+\frac{x_t}{w_t})$, $c_2\ln(1+\frac{\overline{y}_t}{w_t})$ models decreasing marginal effectiveness of the investments, and the results of the following analysis hold also with this choice).

3.3.1 Analysis of the (rewritten) models

In order to study the analytical properties, in the following we rewrite the model with investment coefficients with a transformation of variables. In this way we can bring back the problem to classical theorems of stochastic control. In particular we demonstrate that the HJB equation has solution, unique in a space of polynomial growth functions, and this solution, under suitable hypotheses, is equal to the value function.
Let us consider the transformation of variables

$$\begin{cases} z_t = \frac{x_t}{w_t}\,, \\ v_t = \ln(w_t)\,. \end{cases} \tag{3.3}$$

The expected well-being becomes

$$G(v_t, z_t) = F(e^{v_t}, z_t e^{v_t})$$

$$= (1-p)a\ln(e^{v_t} - \theta z_t e^{v_t}) + pa\ln(\, e^{v_t} - \theta z_t e^{v_t} - f\theta(e^{v_t} - z_t e^{v_t})\,) + b\frac{\alpha}{\theta}\ln(l + z_t e^{v_t})$$

$$= (1-p)a[v_t + \ln(1-\theta z_t)] + pa[v_t + \ln(\,1 - \theta z_t - f\theta(1-z_t)\,)] + b\frac{\alpha}{\theta}\ln(l + z_t e^{v_t})\,.$$

Applying Ito's formula, we obtain

$$dv_t = d\ln(w_t) = \frac{1}{w_t}[m + c_1\frac{x_t}{w_t} + c_2\frac{\overline{y}_t}{w_t}]w_t dt - \frac{1}{2}\frac{1}{w_t^2}\sigma^2 w_t^2 dt + \frac{1}{w_t}\sigma w_t dB_t$$

$$= [m + c_1 z_t + c_2\frac{\overline{y}_t}{e^{v_t}} - \frac{1}{2}\sigma^2]dt + \sigma dB_t\,,$$

with

$$\overline{y}_t = (1-p)(e^{v_t} - \theta z_t e^{v_t}) + p(\, e^{v_t} - \theta z_t e^{v_t} - f\theta(e^{v_t} - z_t e^{v_t})\,)\,.$$

Let us observe that we used Ito's formula without the hypotheses of Ito's lemma (because the logarithmic function is defined only on positive numbers). But if a process v_t verifies this stochastic differential law, then e^{v_t} verifies the law (3.2) (indeed, this time we can apply Ito's lemma, because the exponential function is defined on the whole $\mathbb{R}$).
By using the transformation of variables (3.3), therefore, the stochastic control problem becomes

$$
\begin{array}{l}
\sup_{z\in A'} E_0\Big(\int_0^T G(v_t, z_t)dt\Big)\,, \\[2ex]
dv_t = [m + c_1 z_t + c_2 \frac{\overline{y}_t}{e^{v_t}} - \frac{1}{2}\sigma^2]dt + \sigma dB_t\,,
\end{array}
\tag{3.4}
$$

where A' is the set of the admissible controls, defined as

$$
A' = \left\{
\begin{array}{l}
z(t,\omega) : [0,T]\times\Omega \to \mathbb{R}\; such\, that : \\
1)\; z_t(\omega)\; is\, measurable\, with\, respect\, to\, \mathscr{F}_t\, and\, \mathscr{B}(\mathbb{R}),\; for\, all\, t. \\
2)\; There\, exists\, a\, Markov\, process\, v\, solution\, of\, the\, stochastic \\
differential\, law,\; pathwise\, unique. \\
3)\; 0 \le z_t \le 1\; (in\, [0,T]\times\Omega). \\
4)\; E_0\Big(\int_0^T G(v_t, z_t)dt\Big)\; exists. \\
5)\; z\, is\, progressively\, measurable.
\end{array}
\right\}
$$

(for technical reasons, now we require that the admissible controls are progressively measurable).
The value function of the problem is

$$
R(v_t, t) = \sup_{z\in A'} E_t\Big(\int_t^T G(v_s, z_s)ds\Big)\,,
$$

and the related HJB equation is

$$
\begin{array}{ll}
\sup_{z\,(with\, 0\le z\le 1)} & \Big[G(v,z) + R_v(v,t)[m + c_1 z + c_2 \frac{\overline{y}}{e^v} - \frac{1}{2}\sigma^2] \\
& +\frac{1}{2}R_{vv}(v,t)\sigma^2 + R_t(v,t)\Big] = 0\,,
\end{array}
\tag{3.5}
$$

with terminal condition

$$
R(v,T) = 0\,. \tag{3.6}
$$

The following theorem summarizes some important properties related to the HJB equation and to the solution of the stochastic control problem.

Theorem 1.

1. *There exists a solution $\overline{R}(v,t)$ of the HJB equation (3.5) (with terminal condition (3.6)), unique in the space of functions $C^{2,1}(\mathbb{R}\times[0,T]) \cap C_p(\mathbb{R}\times[0,T])$ ($C_p(\mathbb{R}\times[0,T])$ is the set of continuous functions ϕ on $\mathbb{R}\times[0,T]$, satisfying the polynomial growth condition $|\phi(v,t)| \le k(1+|v|^h)$ for some constants k, h).*

2. We have, for all v, t,

$$\overline{R}(v,t) \geq R(v,t)$$

(namely, the solution of the HJB equation is greater than or equal to the value function).

Furthermore, if there exists an admissible control process $\overline{z}_t$ that (for almost all $(t,\omega) \in [0,T] \times \Omega$) maximizes

$$G(\overline{v}_t, z) + \overline{R}_v(\overline{v}_t, t)[m + c_1 z + c_2 \frac{\overline{y}_t}{e^{\overline{v}_t}} - \frac{1}{2}\sigma^2]$$

(with $\overline{y}_t = (1-p)(e^{\overline{v}_t} - \theta z e^{\overline{v}_t}) + p(\, e^{\overline{v}_t} - \theta z e^{\overline{v}_t} - f\theta(e^{\overline{v}_t} - z e^{\overline{v}_t})\,)$ and $\overline{v}_t$ the state process related to $\overline{z}_t$), then, for all v, t,

$$\overline{R}(v,t) = R(v,t)$$

(namely, the solution of the HJB equation is equal to the value function) and $\overline{z}_t$ is an optimal control process.

Proof. In order to demonstrate the first part of the theorem, let us observe that the model (3.4) verifies the hypotheses of an existence theorem for the solutions of the HJB equation, with uniqueness in the space of (polynomial growth) functions $C^{2,1}(\mathbb{R} \times [0,T]) \cap C_p(\mathbb{R} \times [0,T])$ (see Appendix A.4, Theorem 4). About the hypotheses, in particular, we can observe that the HJB equation is uniformly parabolic (because $\sigma^2 > 0$), G, G_v have polynomial growth (with respect to v and z), and the control processes have values in a compact set ($[0,1]$).
The second part follows from a verification theorem (see Appendix A.4, Theorem 3).

□

In order to express the (stochastic) total income $\overline{X}_{Tot}$ that the taxpayer expects to declare, and the related (stochastic) total income W_{Tot}, we can consider the following equalities

$$\overline{X}_{Tot} = \int_0^T \overline{z}_t e^{v_t} dt \,,$$

$$W_{Tot} = \int_0^T e^{v_t} dt \,,$$

obtained inverting the transformation of variables (3.3), with $\overline{z}_t$ optimal control process for the problem (3.4), and v_t the related state process.

3.3.2 Computational example

In the following example we assign values to the parameters of the (rewritten) model (for simplicity, in the case with null confidence). Solving the stochastic control problem (3.4) and using a Monte Carlo method, the mean and the variance of $\overline{X}_{Tot}$ are estimated.
Let us assume

$$p = 0.15\,,\ f = 1.8\,,\ \theta = 0.45\,,\ w_0 = 1\,,\ a = 1\,,\ b = 0\,,$$

$$m = 0.2\,,\ c_1 = 0.5\,,\ c_2 = 0.5\,,\ \sigma = 1\,,\ T = 1\,.$$

With a suitable algorithm, the maximization problem in the HJB equation (3.5) is solved and the value function (as a solution of the corresponding partial differential equation, numerically solved) is estimated (see figure 3.1). Then the algorithm simulates $10 \times n$ random variables with standard normal distribution and it determines n (discretized) paths of v_t (each consisting of 10 values in equally spaced times, from $t = 0.1$ to $t = 1$), with the corresponding n paths of x_t. For each path of x_t it determines, with numerical integration, the corresponding $\overline{X}_{Tot}$. Finally, with a Monte Carlo method, the algorithm calculates (as in the examples of the previous chapter) the estimator $M = \frac{\sum_{j=1}^{n} \overline{X}_{Tot_j}}{n}$ for the mean of $\overline{X}_{Tot}$, the estimator $S^2 = \frac{(\sum_{j=1}^{n} \overline{X}^2_{Tot_j}) - nM^2}{n-1}$ for the variance and an approximate confidence interval (at the level of 95%) $I = \left[M - 1.96\sqrt{\frac{S^2}{n}}, M + 1.96\sqrt{\frac{S^2}{n}}\right]$.
The code of the algorithm is contained in the attachments 1 and 2 (with an introduction in Appendix C).
Executing the algorithm with $n = 10^4$ simulations of paths, it was obtained

$$M = 0.182709\,,\ S^2 = 0.00541982\,,\ I = [0.181266, 0.184152]\,.$$

Varying the parameters of the problem, the following tables were obtained. In the first table there are the values of the estimated mean M, varying a and c_1.

c_1 / a	0.25	0.5	0.75	1	1.25
1	0.000	0.183	0.553	0.954	1.325
1.5	0.000	0.183	0.560	0.947	1.303

We can observe that, increasing the coefficient of public investment c_1, the (expected) total declared income tends to increase (in both cases $a = 1$ and $a = 1.5$).

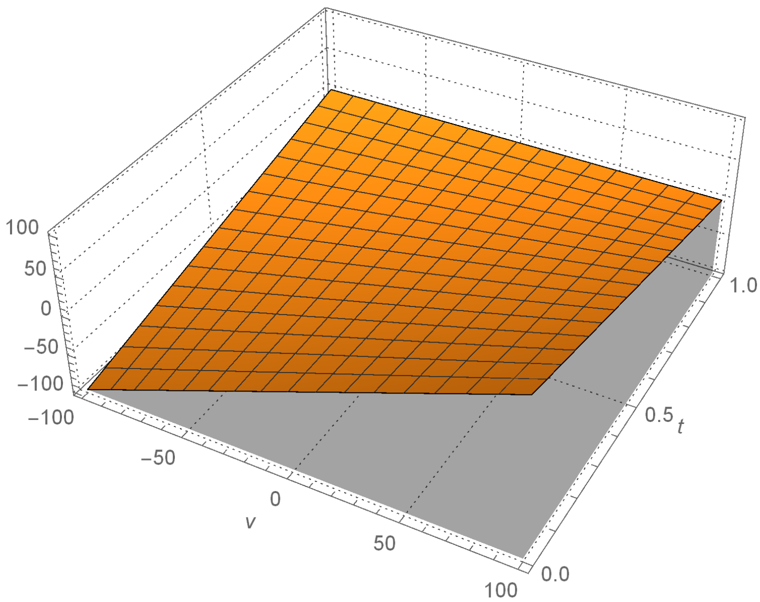

Figure 3.1: The value function $V(v,t)$.

In the second table there are the values of the estimated mean M, varying p and f.

f / p	1.6	1.7	1.8
0.14	0.030	0.072	0.146
0.15	0.045	0.097	0.183
0.16	0.063	0.125	0.224

As we can see, increasing the probability of being discovered p or the fine f, the (expected) total declared income tends to increase.
In the third table there are the values of the estimated mean M, varying p and c_2.

c_2 / p	0.25	0.5	0.75
0.14	0.237	0.146	0.064
0.15	0.274	0.183	0.097
0.16	0.315	0.224	0.136

We can observe that the (expected) total declared income tends to increase as the probability of being discovered p increases. On the other hand, the (expected) total declared income tends to decrease as the coefficient of private investment c_2 increases.

In the fourth table there are the values of the estimated mean M, varying σ and θ.

θ / σ	0.35	0.4	0.45
1	0.290	0.189	0.183
1.5	0.289	0.190	0.184
2	0.290	0.191	0.183

As we can see, increasing the tax rate θ, the (expected) total declared income tends to decrease. Furthermore, varying σ, the (expected) total declared income seems to stay rather stable.
In the fifth table there are the values of the estimated variance S^2, varying w_0 and σ.

σ / w$_0$	1	1.25	1.5	1.75	2
1	0.005	0.009	0.016	0.023	0.033
e$^{\frac{1}{2}}$	0.015	0.027	0.046	0.060	0.094

We can observe that, increasing the constant σ, the variance of the total declared income tends to increase (in both cases $w_0 = 1$ and $w_0 = e^{\frac{1}{2}}$).

3.4 Conclusions of this chapter

In this chapter we propose a general stochastic control model on the expectations of a taxpayer (or, in an alternative interpretation, forecasting the taxpayer's behavior), in which the dynamics of income are driven by economic factors linked to declared income (like public and private investments) and by sources of uncertainty. As a particular case (weakening the hypotheses on U), we consider a model that allows to solve a paradox in the taxpayer's choice with risk neutrality. Then we consider a class of models with public and private investment coefficients, that we study in their analytical properties with a suitable transformation of variables, offering also a computational example.

Some developments can regard the deepening of the mathematical properties of the models (for instance, in the case of models with investment coefficients, the possibility of using constant relative risk aversion utilities, different from the logarithmic function). Other developments can regard the use of these models as part of general equilibrium models.
Furthermore, the parameters can be calibrated with real data, also in order to check the effectiveness of the model results.

Conclusions

In Chapter 1 we propose a static model on tax evasion with confidence in institutions. Thanks to the confidence function added to the utility of income, the taxpayer may choose tax compliance even in case of convenient gamble. Furthermore, an increase in tax rate may lead to an increase in tax evasion, therefore this model proposes a solution to Yitzhaki's paradox. Within our model we can also describe different types of taxpayers.

Then, in Chapter 2 we present two models (the first in discrete time, the second in continuous time) on the taxpayer's expectations (or, in an alternative interpretation, forecasting the taxpayer's behavior).
We study the cases of only one choice, n discrete time choices, and continuous time choices, also under the additional assumption of inflation.
The structure of these two models allows us to study their solutions relying on the static model of the previous chapter. We show various examples in which, using Monte Carlo methods, the models are studied under hypotheses of probability distribution for the stochastic variables representing the parameters.

Finally, in Chapter 3 we propose a stochastic control model that extends the previous models, taking into account dynamics of public and private investments.
By using the dynamic programming principle, we heuristically derive the Hamilton-Jacobi-Bellman equation of the general model. Then we pass to examine two particular cases. The first, in a deterministic framework, solves a paradox (that arises in a static model with linear utility and identically zero confidence function) for the risk neutral taxpayer. In the second case, specifying the form of the investment functions, we analytically study the links between the value function of the (rewritten) problem and an existing solution of the HJB equation, and we use these properties to solve the stochastic control problem with a numerical algorithm.

All the three chapters present a conclusions section, where the possible developments of the models are discussed. More in general, these models can be extended in their economic features and deepened in their mathematical properties. They can also be part of a general equilibrium model. In order to verify assumptions and results of our models, they can be compared with data and results of empirical studies (see also, in Appendix B, a verification project for the assumptions of the static model).

Appendix A

Specifications on the models

A.1 Hypothesis on f

The hypothesis $f < \frac{1}{\theta}$ ensures that even if the evader is discovered his or her effective income Z remains positive (in this way we avoid dealing with the utility of negative amounts, with respect to which the hypothesis of concavity is problematic). Indeed (since $0 < \theta < 1$, $W > 0$ and $0 \leq X \leq W$) we have

$$-\theta X \geq -X$$

and

$$-f\theta(W - X) \geq -(W - X)$$

(where at least one of the two inequalities is strict), therefore

$$Z = W - \theta X - f\theta(W - X) > W - X - (W - X) = 0\,.$$

The hypothesis is also necessary to have $Z > 0$ for all X. Indeed if we assume $f \geq \frac{1}{\theta}$, for $X = 0$ we have

$$Z = W - \theta X - f\theta(W - X) = W - f\theta W = (1 - f\theta)W \leq 0\,.$$

A.2 Derivative of $\overline{X}$ with respect to θ

Theorem 2. *Let $\overline{X}$ be the interior solution that maximizes $E[B]$ for a certain value $\widehat{\theta}$ of the tax rate (fixed the other parameters). Then there exists a neighbourhood I of $\widehat{\theta}$ in which the maximization problem still admits interior* *solution (thus defining the maximum function $\overline{X}(\theta)$ in I) and, furthermore, the maximum function is continuously differentiable and one has (1.11).*

Proof. Let us consider the function of θ, X defined by the first order condition (1.4). Thanks to the regularity of the functions and to the sign of the second order condition (1.5), the assumptions of the implicit function theorem are satisfied in the point $(\widehat{\theta}, \overline{X})$. Thus there exists, continuously differentiable, the implicit function $\overline{X}(\theta)$ in a suitable neighbourhood I of $\widehat{\theta}$. Choosing I suitably small (so that the function $\overline{X}(\theta)$, by continuity, takes values in $]0, W[$), $\overline{X}(\theta)$ is the maximum function sought.
Notice that the implicit function theorem gives directly the form of $\frac{\partial \overline{X}}{\partial \theta}$. Nevertheless here we get it explicitly. Let us observe that the points $(\theta, \overline{X}(\theta))$ meet the first order condition, also in the equivalent form (1.10). Therefore the left-hand side of (1.10), for $X = \overline{X}(\theta)$, is a constant function (equal to 0) of θ in I, then it has null derivative. Differentiating the left-hand side of (1.10) with respect to θ and setting

$$\overline{Y} = W - \theta \overline{X}\,,$$

$$\overline{Z} = W - \theta \overline{X} - f\theta(W - \overline{X})\,,$$

we obtain

$$(1-p)\Big(-\overline{X}-\theta\frac{\partial \overline{X}}{\partial \theta}\Big)U''(\overline{Y})+(1-f)p\Big(-\overline{X}-\theta\frac{\partial \overline{X}}{\partial \theta}-f(W-\overline{X})+f\theta\frac{\partial \overline{X}}{\partial \theta}\Big)U''(\overline{Z})$$

$$+\frac{-\theta\Big(C_{y_2y_2}(\overline{X},\theta,\alpha)\frac{\partial \overline{X}}{\partial \theta}+C_{y_2\theta}(\overline{X},\theta,\alpha)\Big)+C_{y_2}(\overline{X},\theta,\alpha)}{\theta^2}=0\,,$$

that is equivalent to

$$\frac{\partial \overline{X}}{\partial \theta}\Big((p-1)\theta U''(\overline{Y})+(f-1)p\theta U''(\overline{Z})-f(f-1)p\theta U''(\overline{Z})-\frac{C_{y_2y_2}(\overline{X},\theta,\alpha)}{\theta}\Big)$$

$$+(p-1)\overline{X}U''(\overline{Y})+(f-1)p\overline{X}U''(\overline{Z})+f(f-1)p(W-\overline{X})U''(\overline{Z})$$

$$-\frac{C_{y_2\theta}(\overline{X},\theta,\alpha)}{\theta}+\frac{C_{y_2}(\overline{X},\theta,\alpha)}{\theta^2}=0\,,$$

that is

$$\frac{\partial \overline{X}}{\partial \theta}\Big((p-1)\theta U''(\overline{Y})+p\theta(f-1-f^2+f)U''(\overline{Z})-\frac{C_{y_2y_2}(\overline{X},\theta,\alpha)}{\theta}\Big)$$

$$=-\overline{X}\Big((p-1)U''(\overline{Y})+(f-1)pU''(\overline{Z})\Big)-f(f-1)p(W-\overline{X})U''(\overline{Z})+$$

$$+\frac{C_{y_2\theta}(\overline{X},\theta,\alpha)}{\theta}-\frac{C_{y_2}(\overline{X},\theta,\alpha)}{\theta^2}\,.$$

Recalling that the amount

$$D=\theta^2(1-p)U''(\overline{Y})+\theta^2p(f-1)^2U''(\overline{Z})+C_{y_2y_2}(\overline{X},\theta,\alpha)$$

is non-zero (being the first member of the second order condition), we obtain (1.11)

$$\frac{\partial \overline{X}}{\partial \theta}=\frac{-\theta}{D}\Big[\overline{X}\Big((1-p)U''(\overline{Y})-p(f-1)U''(\overline{Z})\Big)-pf(f-1)(W-\overline{X})U''(\overline{Z})+$$

$$+\frac{C_{y_2\theta}(\overline{X},\theta,\alpha)}{\theta}-\frac{C_{y_2}(\overline{X},\theta,\alpha)}{\theta^2}\Big]\,.$$

□

A.3 An elementary comparison result

We use this comparison result in the optimal choice of Section 4.2. The proof is very easy, and we report it only for completeness.

Proposition 4. *Consider $F,G:[0,1]\to\mathbb{R}$, and $\phi,\psi:[0,1]\to\mathbb{R}$ solutions (respectively) of $d\phi=F(t)dt$, $d\psi=G(t)dt$, with $\phi(0)=\psi(0)$. If $F(t)\geq G(t)$ for all $t\in[0,1]$, then $\phi(t)\geq\psi(t)$ for all $t\in[0,1]$.*

Proof. We have, for all $t\in[0,1]$,

$$\phi(t)=\phi(0)+\phi(t)-\phi(0)=\phi(0)+\int_0^t F(s)ds$$

$$\geq\psi(0)+\int_0^t G(s)ds=\psi(0)+\psi(t)-\psi(0)=\psi(t)\,.$$

□

A.4 Stochastic control theorems

The following verification theorem is a particular case of Theorem 3.1 in Fleming and Soner, 2006, p. 157. We use a different notation, similar to that used in our model.

Let U be a compact subset of $\mathbb{R}$. Let $f(v,t,z)$ be a real function of class $C^0(\mathbb{R}\times[0,T]\times U)\cap C^1(\mathbb{R}\times[0,T])$, $\sigma(v,t)$ a real function of class $C^1(\mathbb{R}\times[0,T])$, $G(v,t,z)$ a real function of class $C^0(\mathbb{R}\times[0,T]\times U)$.
Let us suppose that:

1. There exists a real constant C_1 such that $|f_t|+|f_v|\leq C_1$, $|\sigma_t|+|\sigma_v|\leq C_1$, $|f(v,t,z)|\leq C_1(1+|v|+|z|)$, $|\sigma(v,t)|\leq C_1(1+|v|)$.
2. There exist real constants C_2, k such that $|G(v,t,z)|\leq C_2(1+|v|^k+|z|^k)$ (polynomial growth condition).

Let us consider the stochastic control problem, in the form of a value function,

$$R(v_t,t)=\sup_{z\in A'}E_t\Big(\int_t^T G(v_s,t,z_s)ds\Big)\,,$$

$$dv_t=f(v,t,z)dt+\sigma(v,t)dB_t\,,$$

with initial value v_0, B_t a Brownian motion in a probability space $(\Omega,\mathscr{F},P)$ with filtration $\{\mathscr{F}_t\}$, A' the set of admissible controls (namely the progressively measurable processes z_t with values in U), associated HJB equation

$$\sup_{z\in U}\Big[G(v,t,z)+R_v(v,t)f(v,t,z)+\frac{1}{2}R_{vv}(v,t)\sigma^2(v,t)+R_t(v,t)\Big]=0$$

with terminal condition $R(v,T)=0$.
Let $C_p(\mathbb{R}\times[0,T])$ be the set of continuous functions ϕ on $\mathbb{R}\times[0,T]$, satisfying the polynomial growth condition $|\phi(v,t)|\leq k(1+|v|^h)$ for some real constants k, h.

Theorem 3. *If the HJB equation has a solution $\overline{R}(v,t)$ in the space of functions $C^{2,1}(\mathbb{R}\times[0,T])\cap C_p(\mathbb{R}\times[0,T])$, then, for all v,t,*

$$\overline{R}(v,t)\geq R(v,t)$$

(namely, the solution of the HJB equation is greater than or equal to the value function).

Furthermore, if there exists an admissible control process $\overline{z}_t$ that (for almost all $(t, \omega) \in [0, T] \times \Omega$) maximizes $G(\overline{v}_t, t, z) + \overline{R}_v(\overline{v}_t, t) f(\overline{v}_t, t, z)$ ($\overline{v}_t$ is the state process related to $\overline{z}_t$), then, for all v, t,

$$\overline{R}(v, t) = R(v, t)$$

(namely, the solution of the HJB equation is equal to the value function) and $\overline{z}_t$ is an optimal control process.

The following theorem of existence and uniqueness is a particular case of Theorem 4.3 in Fleming and Soner, 2006, p. 163 (see also Theorem 6.2 in Fleming and Rishel, 1975, p.169). Also in this case we use a different notation, similar to that used in our model.

Theorem 4. *Let us consider the equation (of the HJB type)*

$$\sup_{z \in U} \left[G(v, t, z) + R_v(v, t) f(v, t, z) + \frac{1}{2} R_{vv}(v, t) \sigma^2(v, t) + R_t(v, t) \right] = 0 \,, \quad \text{(A.1)}$$

with terminal condition $R(v, T) = 0$, where G is a real function of class C^1 in $\mathbb{R} \times [0, T] \times U$, U is a compact subset of $\mathbb{R}$, f is a real function defined in $\mathbb{R} \times [0, T] \times U$, σ is a real function of class $C^{2,1}$ in $\mathbb{R} \times [0, T]$.
Let us suppose that:

1. *There exists $c > 0$ such that $\sigma^2(v, t) \geq c$ for all v, t (condition of uniformly parabolic HJB equation).*
2. *$f(v, t, z) = \sigma(v, t) \Psi(v, t, z)$ with Ψ a real function of class C^1 in $\mathbb{R} \times [0, T] \times U$ and Ψ, Ψ_v bounded.*
3. *$\sigma, \frac{1}{\sigma}, \sigma_v$ are bounded.*
4. *G, G_v have polynomial growth with respect to v and z.*

Then, there exists a solution $\overline{R}(v, t)$ of (A.1), unique in the space of functions $C^{2,1}(\mathbb{R} \times [0, T]) \cap C_p(\mathbb{R} \times [0, T])$.

Appendix B

A verification project for the assumptions of the static model

The static model of Chapter 1 (on tax evasion and confidence in institutions) assumes some hypotheses that are original compared to the reference literature. We wish to submit these assumptions to qualitative and quantitative analysis, in order to understand if they can have an empirical evidence. In particular, in the following are listed three hypotheses to test.
1) For all y_2, θ, α we have

$$C_{y_2}(y_2, \theta, \alpha) \geq 0\,,$$

namely, the confidence function is monotonically non-decreasing with respect to declared income.
2) There exists $\overline{\theta} \in [0, \frac{1}{f}[$ such that for all $\theta > \overline{\theta}$ (and for all y_2, α) we have

$$C_{y_2\theta}(y_2, \theta, \alpha) \leq 0\,,$$

that is, the marginal confidence of declared income is monotonically non-increasing with respect to θ, at least above $\overline{\theta}$.
3) For all y_2, θ, α we have

$$C_{y_2\alpha}(y_2, \theta, \alpha) \geq 0\,,$$

that is, the marginal confidence of declared income is monotonically non-decreasing with respect to α.

We propose a verification project for these assumptions, based on interviews (in order to have case studies, that allow to know the dynamics of the taxpayer's choices) and questionnaires (in order to conduct an econometric analysis).

The use of mixed methods (see Creswell, 2013), based on qualitative and quantitative analysis, allows to study different aspects of the phenomenon, and the results can be integrated in order to have a more complete perspective on the reliability of the hypotheses.

B.1 Interviews (qualitative verification)

In order to understand in depth the meanings linked to the concept of confidence (trust in institutions, social responsibility, utility of contributing to the collective welfare) and the connections between tax compliance, tax rate and (perceived) effectiveness of public expenditure, we can use interviews as qualitative methods to open the black box. The interviews can be structured (namely, in each interview there are the same questions), with open-ended answers.
For instance, we can interview 5 people, among entrepreneurs and professionals, with small, medium or large businesses (varying could be better). The interview can be composed of 3 open questions like:

1. If the institutions are trustworthy, can paying taxes be (as well as a duty imposed by law) a way of contributing to the collective welfare? Why?

2. When are taxes too high or unfair?

3. When services work better, are taxes more gladly paid, or this aspect is unimportant? Why?

B.2 Questionnaires (quantitative verification)

In order to verify statistically the hypotheses of the static model on tax evasion and confidence in institutions, we can collect data with a survey based on questionnaires.
It should be proposed to a statistical sample of taxpayers. The key questions, with 3 closed-ended and ordered answers, could be:

1. Do you think that one can feel himself or herself useful paying taxes, as a contribution to the collective welfare? [Highly, a little, in no way].

2. This feeling of 'being useful' is negatively affected, when one perceives that taxes are too high? [Highly, a little, in no way].

3. This feeling of ‘being useful’ is positively affected, when one perceives that the public expenditure achieves good outcomes? [Highly, a little, in no way].

The sample selection, the way to propose and implement the questionnaire, its structure, the additional questions, and the other technical requirements, can be chosen according to suitability and resources.

Appendix C

Code of the algorithm in Section 3.3.2

The attachments 1 and 2 contain, respectively, the first and the second part of the code of the algorithm used in Section 3.3.2. It was implemented and executed in Mathematica 10.3.

The first part of the code (see Attachment 1) solves, with symbolic calculus, the maximization problem in the HJB equation (3.5).

In the second part of the algorithm (see Attachment 2), the partial differential equation (obtained substituting the maximum function of the previous part in the HJB equation) is numerically solved. The solution constitutes an estimate of the value function (the algorithm offers also the graph in figure 3.1). Then $10 \times n$ random variables with standard normal distribution are simulated (with $n = 10^4$), in order to determine n (discretized) paths of v_t (each consisting of 10 values in equally spaced times, from $t = 0.1$ to $t = 1$), with the corresponding n paths of x_t. For each path of x_t the algorithm determines, with numerical integration (trapezoidal rule), the corresponding $\overline{X}_{Tot}$. Finally, with a Monte Carlo method, the algorithm calculates the estimator $M = \frac{\sum_{j=1}^{n} \overline{X}_{Tot_j}}{n}$ for the mean of $\overline{X}_{Tot}$, the estimator $S^2 = \frac{(\sum_{j=1}^{n} \overline{X}_{Tot_j}^2) - nM^2}{n-1}$ for the variance of $\overline{X}_{Tot}$, an approximate confidence interval (at the level of 95%) $I = \left[M - 1.96\sqrt{\frac{S^2}{n}}, M + 1.96\sqrt{\frac{S^2}{n}}\right]$ for $\overline{X}_{Tot}$, and a histogram of its values. The algorithm calculates also the estimator $M_W = \frac{\sum_{j=1}^{n} W_{Tot_j}}{n}$ for the mean of W_{Tot}.

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In[27]:=
```
h[z_, f_, p_, theta_, a_] :=
  (1 - p) * a * (Log[1 - theta * z]) + p * a * (Log[1 - theta * z - f * theta (1 - z)]);
l[z_, j_, f_, p_, theta_, a_, c1_, c2_] := h[z, f, p, theta, a] +
   j * (c1 * z + c2 * (1 - p) * (- theta * z) + c2 * p * (- theta * z) + c2 * p * f * theta * z);
D[l[z, j, f, p, theta, a, c1, c2], z]
```

Out[29]=

$$\mathrm{j}\ (\mathrm{c1} - \mathrm{c2}\ (1 - \mathrm{p})\ \mathrm{theta} - \mathrm{c2}\ \mathrm{p}\ \mathrm{theta} + \mathrm{c2}\ \mathrm{f}\ \mathrm{p}\ \mathrm{theta}) - \frac{\mathrm{a}\ (1 - \mathrm{p})\ \mathrm{theta}}{1 - \mathrm{theta}\ \mathrm{z}} + \frac{\mathrm{a}\ \mathrm{p}\ (-\mathrm{theta} + \mathrm{f}\ \mathrm{theta})}{1 - \mathrm{f}\ \mathrm{theta}\ (1 - \mathrm{z}) - \mathrm{theta}\ \mathrm{z}}$$

In[30]:=
```
derivata[z_, j_, f_, p_, theta_, a_, c1_, c2_] :=
  j (c1 - c2 (1 - p) theta - c2 p theta + c2 f p theta) -
   (a (1 - p) theta)/(1 - theta z) + (a p (- theta + f theta))/(1 - f theta (1 - z) - theta z);
```

In[31]:=
```
Solve[derivata[z, j, f, p, theta, a, c1, c2] == 0, z]
```

Out[31]=

$$\begin{aligned}\{\{z \to (&-2\ c1\ j\ theta + c1\ f\ j\ theta + a\ theta^2 - a\ f\ theta^2 + 2\ c2\ j\ theta^2 + \\ &c1\ f\ j\ theta^2 - c2\ f\ j\ theta^2 - 2\ c2\ f\ j\ p\ theta^2 + c2\ f^2\ j\ p\ theta^2 - \\ &c2\ f\ j\ theta^3 + c2\ f^2\ j\ p\ theta^3 - \sqrt{(} ((2\ c1\ j\ theta - c1\ f\ j\ theta - \\ &\quad a\ theta^2 + a\ f\ theta^2 - 2\ c2\ j\ theta^2 - c1\ f\ j\ theta^2 + c2\ f\ j\ theta^2 + \\ &\quad 2\ c2\ f\ j\ p\ theta^2 - c2\ f^2\ j\ p\ theta^2 + c2\ f\ j\ theta^3 - c2\ f^2\ j\ p\ theta^3)^2 - \\ &\quad 4\ (-c1\ j + a\ theta + c2\ j\ theta + c1\ f\ j\ theta - a\ f\ p\ theta - c2\ f\ j\ p\ theta - \\ &\quad a\ f\ theta^2 - c2\ f\ j\ theta^2 + a\ f\ p\ theta^2 + c2\ f^2\ j\ p\ theta^2)\ (-c1\ j\ theta^2 + \\ &\quad c1\ f\ j\ theta^2 + c2\ j\ theta^3 - c2\ f\ j\ theta^3 - c2\ f\ j\ p\ theta^3 + c2\ f^2\ j\ p\ theta^3))) / \\ &(2\ (-c1\ j\ theta^2 + c1\ f\ j\ theta^2 + c2\ j\ theta^3 - c2\ f\ j\ theta^3 - \\ &\quad c2\ f\ j\ p\ theta^3 + c2\ f^2\ j\ p\ theta^3))\},\end{aligned}$$

$$\begin{aligned}\{z \to (&-2\ c1\ j\ theta + c1\ f\ j\ theta + a\ theta^2 - a\ f\ theta^2 + 2\ c2\ j\ theta^2 + \\ &c1\ f\ j\ theta^2 - c2\ f\ j\ theta^2 - 2\ c2\ f\ j\ p\ theta^2 + \\ &c2\ f^2\ j\ p\ theta^2 - c2\ f\ j\ theta^3 + c2\ f^2\ j\ p\ theta^3 + \\ &\sqrt{(} ((2\ c1\ j\ theta - c1\ f\ j\ theta - a\ theta^2 + a\ f\ theta^2 - \\ &\quad 2\ c2\ j\ theta^2 - c1\ f\ j\ theta^2 + c2\ f\ j\ theta^2 + 2\ c2\ f\ j\ p\ theta^2 - \\ &\quad c2\ f^2\ j\ p\ theta^2 + c2\ f\ j\ theta^3 - c2\ f^2\ j\ p\ theta^3)^2 - \\ &\quad 4\ (-c1\ j + a\ theta + c2\ j\ theta + c1\ f\ j\ theta - a\ f\ p\ theta - c2\ f\ j\ p\ theta - \\ &\quad a\ f\ theta^2 - c2\ f\ j\ theta^2 + a\ f\ p\ theta^2 + c2\ f^2\ j\ p\ theta^2)\ (-c1\ j\ theta^2 + \\ &\quad c1\ f\ j\ theta^2 + c2\ j\ theta^3 - c2\ f\ j\ theta^3 - c2\ f\ j\ p\ theta^3 + c2\ f^2\ j\ p\ theta^3))) / \\ &(2\ (-c1\ j\ theta^2 + c1\ f\ j\ theta^2 + c2\ j\ theta^3 - c2\ f\ j\ theta^3 - \\ &\quad c2\ f\ j\ p\ theta^3 + c2\ f^2\ j\ p\ theta^3))\}\}\end{aligned}$$

In[32]:=
```
derivata[0, j, f, p, theta, a, c1, c2]
derivata[1, j, f, p, theta, a, c1, c2]
```

Out[32]= $-a\,(1-p)\,\text{theta}+\dfrac{a\,p\,(-\text{theta}+f\,\text{theta})}{1-f\,\text{theta}}+j\,(c1-c2\,(1-p)\,\text{theta}-c2\,p\,\text{theta}+c2\,f\,p\,\text{theta})$

Out[33]= $-\dfrac{a\,(1-p)\,\text{theta}}{1-\text{theta}}+\dfrac{a\,p\,(-\text{theta}+f\,\text{theta})}{1-\text{theta}}+j\,(c1-c2\,(1-p)\,\text{theta}-c2\,p\,\text{theta}+c2\,f\,p\,\text{theta})$

In[34]:=
```
derivata0[j_] := -a (1 - p) theta +
    a p (-theta + f theta) / (1 - f theta) + j (c1 - c2 (1 - p) theta - c2 p theta + c2 f p theta);
derivata1[j_] := - a (1 - p) theta / (1 - theta) + a p (-theta + f theta) / (1 - theta) +
    j (c1 - c2 (1 - p) theta - c2 p theta + c2 f p theta);
```

In[36]:=
```
z1[j_] := (-2 c1 j theta + c1 f j theta + a theta^2 - a f theta^2 +
      2 c2 j theta^2 + c1 f j theta^2 - c2 f j theta^2 - 2 c2 f j p theta^2 + c2 f^2 j p theta^2 -
      c2 f j theta^3 + c2 f^2 j p theta^3 - Sqrt[((2 c1 j theta - c1 f j theta -
            a theta^2 + a f theta^2 - 2 c2 j theta^2 - c1 f j theta^2 + c2 f j theta^2 +
            2 c2 f j p theta^2 - c2 f^2 j p theta^2 + c2 f j theta^3 - c2 f^2 j p theta^3)^2 -
         4 (-c1 j + a theta + c2 j theta + c1 f j theta - a f p theta - c2 f j p theta -
            a f theta^2 - c2 f j theta^2 + a f p theta^2 + c2 f^2 j p theta^2) (-c1 j theta^2 +
            c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 - c2 f j p theta^3 + c2 f^2 j p theta^3))]) /
    (2 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
        c2 f j p theta^3 + c2 f^2 j p theta^3));
z2[j_] := (-2 c1 j theta + c1 f j theta + a theta^2 - a f theta^2 + 2 c2 j theta^2 +
      c1 f j theta^2 - c2 f j theta^2 - 2 c2 f j p theta^2 + c2 f^2 j p theta^2 -
      c2 f j theta^3 + c2 f^2 j p theta^3 + Sqrt[((2 c1 j theta - c1 f j theta -
            a theta^2 + a f theta^2 - 2 c2 j theta^2 - c1 f j theta^2 + c2 f j theta^2 +
            2 c2 f j p theta^2 - c2 f^2 j p theta^2 + c2 f j theta^3 - c2 f^2 j p theta^3)^2 -
         4 (-c1 j + a theta + c2 j theta + c1 f j theta - a f p theta - c2 f j p theta -
            a f theta^2 - c2 f j theta^2 + a f p theta^2 + c2 f^2 j p theta^2) (-c1 j theta^2 +
            c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 - c2 f j p theta^3 + c2 f^2 j p theta^3))]) /
    (2 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
        c2 f j p theta^3 + c2 f^2 j p theta^3));
```

In[38]:=

```
zetamassimo[j_] := Piecewise[{{0, derivata0[j] ≤ 0},
    {1, derivata1[j] ≥ 0}, {z1[j], 0 < z1[j] < 1}, {z2[j], 0 < z2[j] < 1}}]
```

```
f = 1.8;
p = 0.15;
theta = 9/20;
v0 = 0;
a = 1;
c1 = 0.5;
c2 = 0.5;
m = 0.2;
s = 1;
n = 10^4;

zetamassimo[j_] := Piecewise[{{0, -a (1 - p) theta + (a p (-theta + f theta))/(1 - f theta) +
      j (c1 - c2 (1 - p) theta - c2 p theta + c2 f p theta) ≤ 0}, {1, -(a (1 - p) theta)/(1 - theta) +
      (a p (-theta + f theta))/(1 - theta) + j (c1 - c2 (1 - p) theta - c2 p theta + c2 f p theta) ≥ 0},
   {(-2 c1 j theta + c1 f j theta + a theta^2 - a f theta^2 + 2 c2 j theta^2 + c1 f j theta^2 -
       c2 f j theta^2 - 2 c2 f j p theta^2 + c2 f^2 j p theta^2 - c2 f j theta^3 +
       c2 f^2 j p theta^3 - Sqrt[((2 c1 j theta - c1 f j theta - a theta^2 + a f theta^2 -
              2 c2 j theta^2 - c1 f j theta^2 + c2 f j theta^2 + 2 c2 f j p theta^2 -
              c2 f^2 j p theta^2 + c2 f j theta^3 - c2 f^2 j p theta^3)^2 -
           4 (-c1 j + a theta + c2 j theta + c1 f j theta - a f p theta - c2 f j p theta -
              a f theta^2 - c2 f j theta^2 + a f p theta^2 + c2 f^2 j p theta^2) (-c1 j theta^2 + c1 f
               j theta^2 + c2 j theta^3 - c2 f j theta^3 - c2 f j p theta^3 + c2 f^2 j p theta^3))])/
     (2 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
         c2 f j p theta^3 + c2 f^2 j p theta^3)),
    0 < (-2 c1 j theta + c1 f j theta + a theta^2 - a f theta^2 + 2 c2 j theta^2 +
         c1 f j theta^2 - c2 f j theta^2 - 2 c2 f j p theta^2 + c2 f^2 j p theta^2 -
         c2 f j theta^3 + c2 f^2 j p theta^3 - Sqrt[((2 c1 j theta - c1 f j theta - a theta^2 + a
                f theta^2 - 2 c2 j theta^2 - c1 f j theta^2 + c2 f j theta^2 + 2 c2 f j
                p theta^2 - c2 f^2 j p theta^2 + c2 f j theta^3 - c2 f^2 j p theta^3)^2 -
             4 (-c1 j + a theta + c2 j theta + c1 f j theta - a f p theta - c2 f j p theta - a
                f theta^2 - c2 f j theta^2 + a f p theta^2 + c2 f^2 j p theta^2) (-c1 j theta^2 + c1 f
                j theta^2 + c2 j theta^3 - c2 f j theta^3 - c2 f j p theta^3 + c2 f^2 j p theta^3))])/
      (2 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
           c2 f j p theta^3 + c2 f^2 j p theta^3)) < 1},
   {(-2 c1 j theta + c1 f j theta + a theta^2 - a f theta^2 + 2 c2 j theta^2 +
       c1 f j theta^2 - c2 f j theta^2 - 2 c2 f j p theta^2 +
       c2 f^2 j p theta^2 - c2 f j theta^3 + c2 f^2 j p theta^3 +
       Sqrt[((2 c1 j theta - c1 f j theta - a theta^2 + a f theta^2 -
              2 c2 j theta^2 - c1 f j theta^2 + c2 f j theta^2 + 2 c2 f j p theta^2 -
```

```
                c2 f^2 j p theta^2 + c2 f j theta^3 - c2 f^2 j p theta^3)^2 -
            4 (-c1 j + a theta + c2 j theta + c1 f j theta - a f p theta - c2 f j p theta -
                a f theta^2 - c2 f j theta^2 + a f p theta^2 + c2 f^2 j p theta^2) (-c1 j theta^2 + c1 f
                 j theta^2 + c2 j theta^3 - c2 f j theta^3 - c2 f j p theta^3 + c2 f^2 j p theta^3)))/
        (2 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
             c2 f j p theta^3 + c2 f^2 j p theta^3)),
       0 < (-2 c1 j theta + c1 f j theta + a theta^2 - a f theta^2 + 2 c2 j theta^2 +
             c1 f j theta^2 - c2 f j theta^2 - 2 c2 f j p theta^2 +
             c2 f^2 j p theta^2 - c2 f j theta^3 + c2 f^2 j p theta^3 +
             √((2 c1 j theta - c1 f j theta - a theta^2 + a f theta^2 - 2 c2 j theta^2 - c1 f j theta^2 +
                   c2 f j theta^2 + 2 c2 f j p theta^2 - c2 f^2 j p theta^2 + c2 f j theta^3 - c2 f^2 j
                    p theta^3)^2 - 4 (-c1 j + a theta + c2 j theta + c1 f j theta - a f p theta -
                   c2 f j p theta - a f theta^2 - c2 f j theta^2 + a f p theta^2 + c2 f^2 j p theta^2)
                 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
                   c2 f j p theta^3 + c2 f^2 j p theta^3)))/
         (2 (-c1 j theta^2 + c1 f j theta^2 + c2 j theta^3 - c2 f j theta^3 -
              c2 f j p theta^3 + c2 f^2 j p theta^3)) < 1}}]
```

```
zetamassimo[1]
```

```
0.472306
```

```
g[v_, z_] :=
  (1 - p) * a * (v + Log[1 - theta * z]) + p * a * (v + Log[1 - theta * z - f * theta (1 - z)]);
```

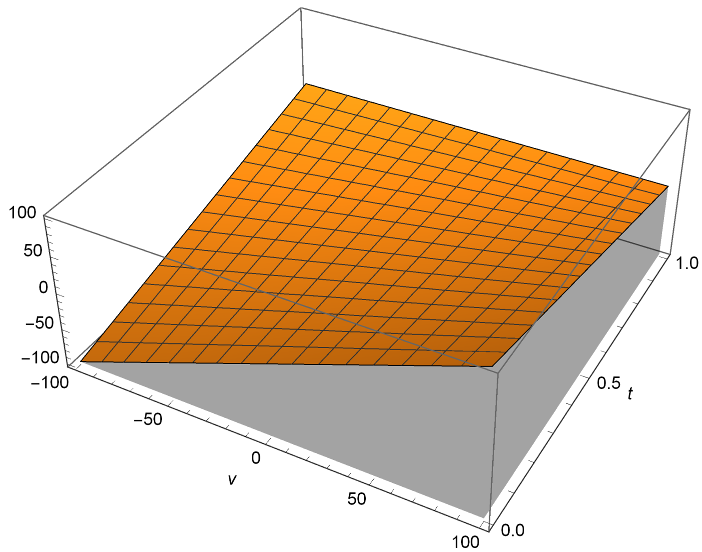
```
fval = NDSolve[{g[v, zetamassimo[D[valore[v, t], v]]] +
      D[valore[v, t], v] * (m + c1 * zetamassimo[D[valore[v, t], v]] +
          c2 * ((1 - p) * (1 - theta * zetamassimo[D[valore[v, t], v]]) +
              p * (1 - theta * zetamassimo[D[valore[v, t], v]] -
                  f * theta (1 - zetamassimo[D[valore[v, t], v]])))) - (1 / 2) * (s^2)) +
      (1 / 2) * D[valore[v, t], v, v] * (s^2) + D[valore[v, t], t] == 0,
    valore[v, 1] == 0 }, valore, {v,
    -100 * (s^2),
    100 * (s^2)}, {t,
    0,
    1}]
```

NDSolve::bcart :
Warning: an insufficient number of boundary conditions have been specified for the direction of independent variable v. Artificial boundary effects may be present in the solution. >>

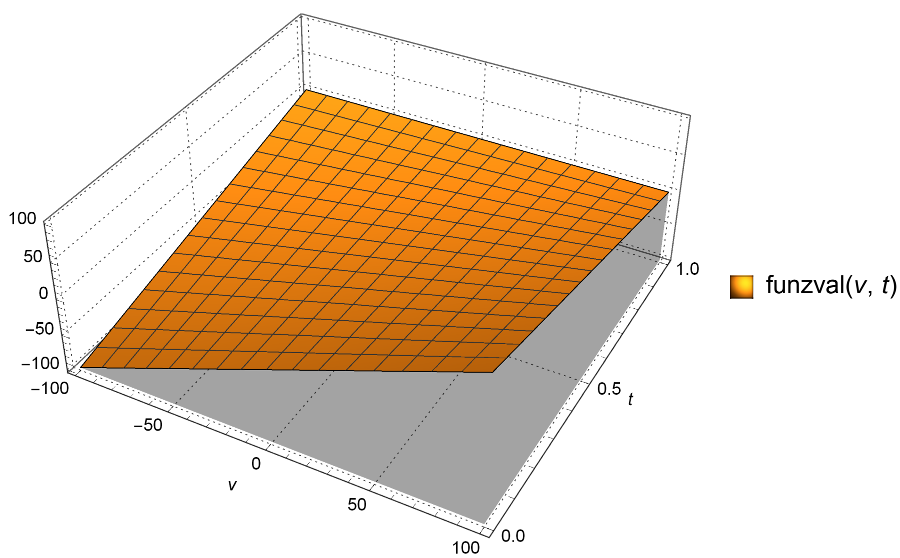
```
{{valore → InterpolatingFunction[ Domain: {{-100., 100.}, {0., 1.}}
                                  Output: scalar ]}}
```

```
funzval[v_, t_] := Evaluate[valore[v, t] /. fval];
```

```
Plot3D[funzval[v, t], {v, -100 * (s^2), 100 * (s^2)},
 {t, 0, 1}, AxesLabel → Automatic, Filling -> Bottom]
```




```
Plot3D[funzval[v, t], {v, -100 s², 100 s²}, {t, 0, 1},
 PlotTheme → "Detailed", AxesLabel → Automatic, Filling → Bottom]
```




```
matv = Table[0, {j, n}, {k, 11}];

matx = Table[0, {j, n}, {k, 11}];

vetwtot = Table[0, {j, n}];

vetxtot = Table[0, {j, n}];
```

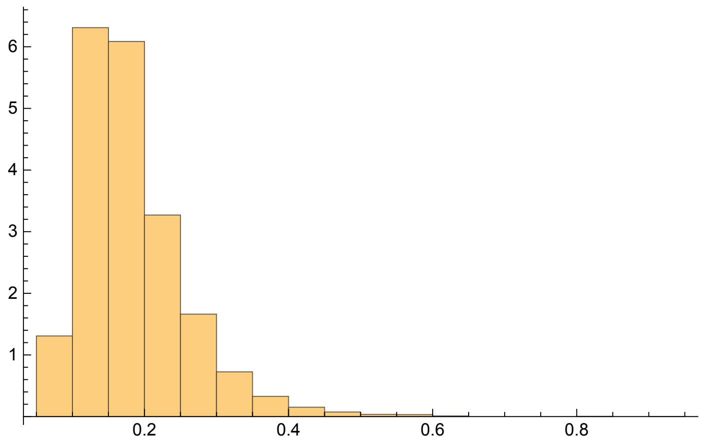
```
aleatoria = RandomVariate[NormalDistribution[0, 1], {n, 10}];
For[j = 1, j ≤ n, j++, matv[[j, 1]] = v0;
  matx[[j, 1]] = Exp[v0] *
    zetamassimo[(funzval[v0 + 10^(-6), 0][[1]] - funzval[v0, 0][[1]]) / (10^(-6))];
  For[k = 2, k ≤ 11, k++, matv[[j, k]] = matv[[j, k - 1]] +
      (m + c1 * zetamassimo[(funzval[matv[[j, k - 1]] + 10^(-6), 0.1 * (k - 2)][[1]] -
             funzval[matv[[j, k - 1]], 0.1 * (k - 2)][[1]]) / (10^(-6))] +
         c2 * ((1 - p) * (1 - theta * zetamassimo[(funzval[matv[[j, k - 1]] + 10^(-6),
                    0.1 * (k - 2)][[1]] - funzval[matv[[j, k - 1]], 0.1 * (k - 2)][[
                   1]]) / (10^(-6))]) + p * (1 - theta * zetamassimo[
                (funzval[matv[[j, k - 1]] + 10^(-6), 0.1 * (k - 2)][[1]] - funzval[
                    matv[[j, k - 1]], 0.1 * (k - 2)][[1]]) / (10^(-6))] - f * theta (1 -
                 zetamassimo[(funzval[matv[[j, k - 1]] + 10^(-6), 0.1 * (k - 2)][[1]] -
                    funzval[matv[[j, k - 1]], 0.1 * (k - 2)][[1]]) / (10^(-6))]))) -
         (1 / 2) * (s^2)) * 0.1 + s * aleatoria[[j, k - 1]] * Sqrt[0.1];
   matx[[j, k]] = Exp[matv[[j, k]]] * zetamassimo[
      (funzval[matv[[j, k]] + 10^(-6), 0.1 * (k - 1)][[1]] -
          funzval[matv[[j, k]], 0.1 * (k - 1)][[1]]) / (10^(-6))]]];
(* We use the trapezoidal rule *)
For[j = 1, j ≤ n, j++,
  vetwtot[[j]] = (1 / 10) * (((Exp[matv[[j, 1]]] + Exp[matv[[j, 11]]]) / 2) +
      Sum[Exp[matv[[j, k]]], {k, 2, 10}])];
mediawtot = (Sum[vetwtot[[j]], {j, 1, n}]) / n
For[j = 1, j ≤ n, j++, vetxtot[[j]] =
   (1 / 10) * (((matx[[j, 1]] + matx[[j, 11]]) / 2) + Sum[matx[[j, k]], {k, 2, 10}])];
mediaxtot = (Sum[vetxtot[[j]], {j, 1, n}]) / n
varianza = (1 / (n - 1)) * (Sum[(vetxtot[[j]])^2, {j, 1, n}] - n * (mediaxtot)^2)
confidenza =
 {mediaxtot - 1.96 * Sqrt[(varianza) / (n)], mediaxtot + 1.96 * Sqrt[(varianza) / (n)]}
Show[Histogram[vetxtot, 20, "ProbabilityDensity"]]
```

```
1.47137

0.182709

0.00541982

{0.181266, 0.184152}
```


6
5
4
3
2
1
0.2
0.4
0.6
0.8